The Open University

S102PRP/S510 MODULE 10 INTO SCIENCE

MODULE 10 SURVEYING

STUDY GUIDE

This Module explores some aspects of surveying, which involves the measurement of distances and angles in order to find the positions of objects relative to each other. For example, the objects might be buildings, roads, the boundaries of property, or something on a grander scale such as the distance between England and France with a view to building the Channel Tunnel. To do this we examine the relationships between the sides and angles of right-angled triangles. The study of these relationships is known as trigonometry.

The other related and important topic that this Module explores is slopes or gradients in relation to roads and railway lines. It also describes the gradients of graphs and introduces the algebraic equation of a straight line graph.

There is no practical work in this Module but there are some 'surveying' exercises that require a pencil, a ruler, a pair of compasses, a protractor and an A4 sheet of graph paper (plain paper will do). A small plastic set square (triangle with a right angle) would also be useful, but your protractor will do instead. We strongly recommend that you have a go at these exercises in order to understand the principles being illustrated. If the concepts covered in this Module are completely new to you, you may find it fairly difficult and you may need more than six hours to study it. We have provided a lot of SAQs so you can practise the tricky mathematical skills covered. It is important that by the end of the Module you have acquired the three skills given in the Overview.

You will need your calculator to hand; make sure that it is set to DEG (degrees) mode for the whole of this Module.

1 INTRODUCING SURVEYING

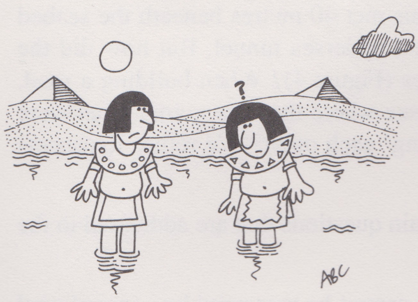

FIGURE 1 Egyptians puzzling over the boundaries of flooded fields.

Surveying has a long history. The Egyptians around 3 000 BC, for example, depended upon the River Nile for water to grow grain, and at that time the river flooded the fields each year so obliterating their boundaries. Surveying techniques were developed for re-establishing the boundaries of fields, calculating area and settling any disputes over land ownership. The Egyptians developed ingenious methods of irrigation for their crops (channelling water from the river). They also built large pyramids, each with a perfectly square base, aligned with one side facing North. Laying out and building these structures took considerable knowledge of what we would call civil engineering. A combination of necessity and civic pride was the incentive for the development of new skills and knowledge of geometry and surveying (Figure 1).

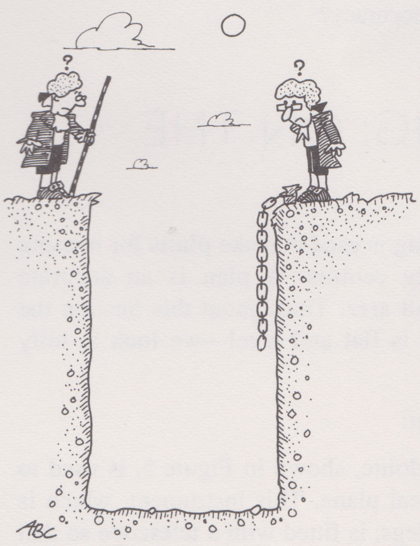

FIGURE 2 A chain too short.

More recently, surveying techniques were used to create accurate maps of Britain and especially to establish distances where direct measurement was not possible, for example across the Avon Gorge, so that Isambard Kingdom Brunel's famous Clifton Suspension Bridge could be built near Bristol during 1836–1864. Similarly, surveying established the distances involved for the bridges across the Menai Straits between Anglesey and North Wales, and in the 1980s for the bridges across the rivers Severn and Humber. In all these cases the objective (the other side) could be clearly seen, but methods of measuring using the surveyor's chain (traditionally 22 yards long) was not possible (Figure 2)!

In the early 19th century scientists decided to establish the exact position of the landmasses of England and France relative to each other. In this way the distance between the two most eminent observatories at the time, London (Greenwich) and Paris, could be calculated. It may be difficult for us in the 1990s, to remember that in 1750 this distance was not known to even the nearest kilometre. At that time, the first suggestions were made for a tunnel beneath the English Channel to link England with France. The hindsight of well-developed engineering technology shows most of the ideas appear rather fanciful and

hazardous to shipping, with central islands and ventilation towers in the Channel itself, as shown in Figure 3. Remember that the planners intended the tunnel to be used by stage-coach traffic, as these ideas pre-dated the railway system which began in the 1830s.

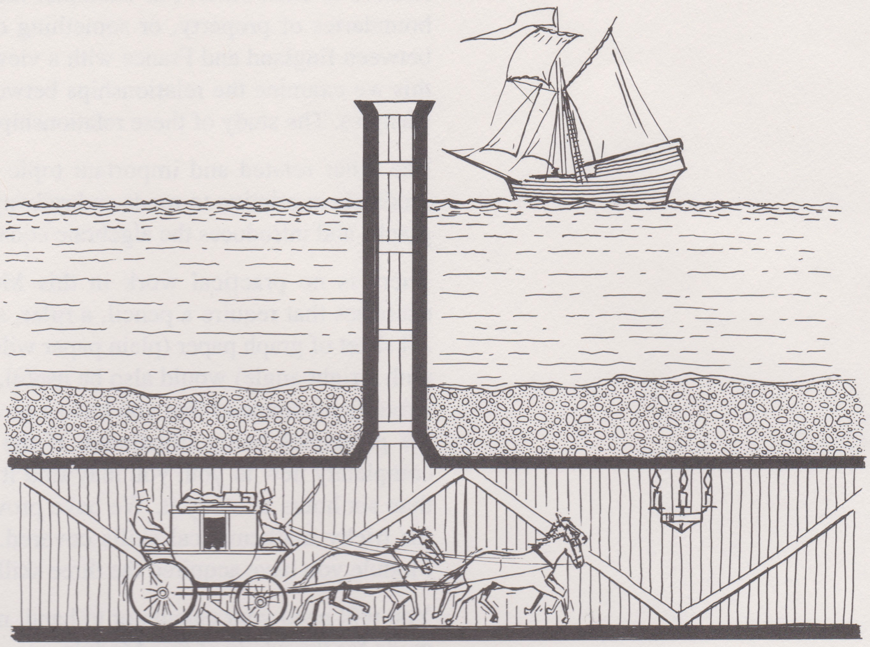

FIGURE 3 Plan of the Channel Tunnel for use by stage coaches, circa 1800, with islands, ventilation tubes and lit by oil lamps.

FIGURE 4 How easy is it for the teams to miss each other?

Two centuries later in 1990, after one of the largest engineering projects of the century, British and French tunnelling teams met 40 metres beneath the seabed of the English Channel, the world's longest undersea tunnel. But how did the tunnellers know how to 'aim' the machines (Figure 4)? When building a road, canal or a bridge the workers can at least see where they are going; what were the chances of the tunnelling teams 'missing' each other? We return to aspects of the Channel Tunnel later in the Module.

This introductory Section has set up two main questions that are addressed in the Module:

1 How can we 'work out' distances that cannot be measured by conventional means?

2 How can surveying be used to ensure that an engineering project, such as a tunnel or a motorway, maintains on-target accuracy?

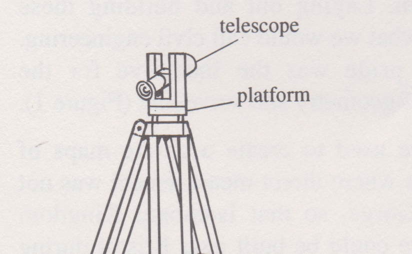

FIGURE 5 A theodolite.

2 SIMPLE SURVEYING: ON THE LEVEL

As a starting point close to home, **surveying** is used to make **plans** for housing estates, roads, factories and community centres. A plan is an accurate representation on paper—a map of a small area. Throughout this Section the assumption has been made that the area is flat and level—we look at hilly ground later in the Module.

Surveyors use two basic types of instrument:

1 An accurate protractor called a theodolite, shown in Figure 5, is used to measure angles in a horizontal or a vertical plane. This instrument, which is mounted on a small platform with tripod legs, is fitted with a telescope so that you can see distant objects. By rotating the scale on the platform the angle between two objects can be read directly. You may have seen people using these

instruments, in association with red and white horizontally striped poles, for example near roads that are scheduled for straightening.

2 A measuring tape is used to measure distances. This has been replaced by devices that measure distances electronically. These devices are basically just tape measures and accurate surveying took place before these tools were invented.

2.1 DISTANCE MEASUREMENT

The aim of this Section is for you to do some 'surveying', by constructing triangles and measuring angles. To do this you use Figure 6 as a basis for the work, and to which you add other features. Redraw Figure 6 on an A4 sheet of graph paper or plain paper, with the lines towards the bottom of the page, using the exact measurements given in Guided Exercise 1. We strongly advise you to do the activities in this Guided Exercise, where you measure angles and lengths of sides of triangles.

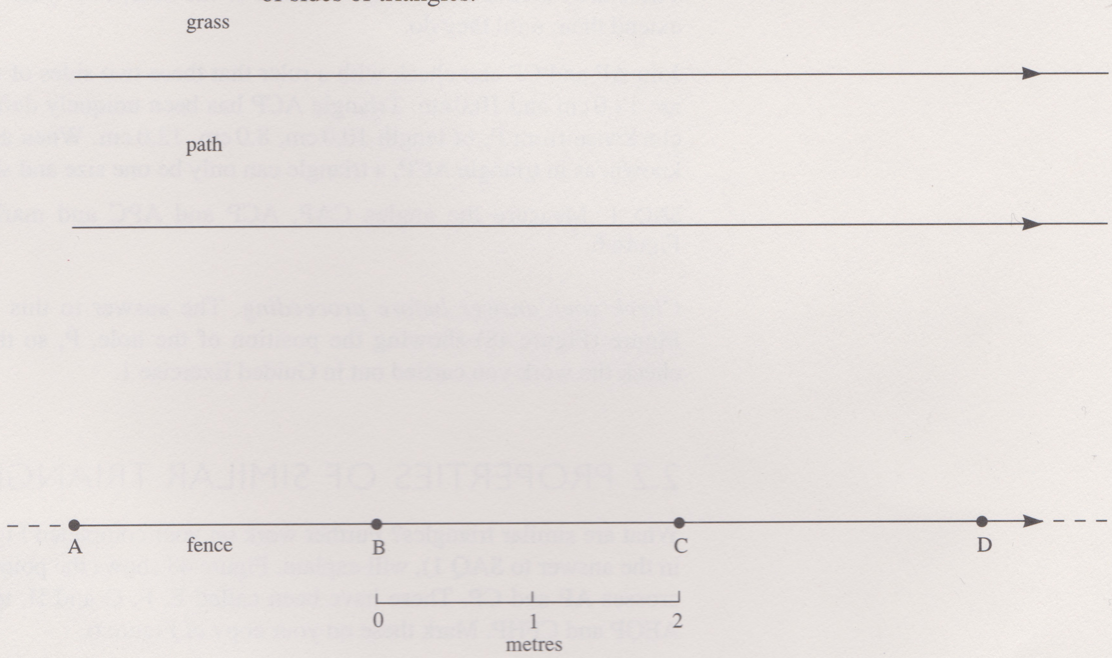

FIGURE 6 The boundary fence and a path in a park.

GUIDED EXERCISE 1: SURVEYING IN THE PARK, USING DISTANCE MEASUREMENT

Our exploration begins in a park and in particular with the features shown in Figure 6. There is a path 1.00 m wide, parallel to and 2.00 m away from a boundary fence that has posts 2.00 m apart and labelled A, B, C and D. The small arrowheads indicate that the sides of the path and the fence are all parallel (Module 9, Section 2.2). The scale bar shows that 1.00 m in the park is represented by 2.00 cm on the plan, so on Figure 6 the path is 2.00 cm wide and 4.00 cm away from the fence, and the posts are 4.00 cm apart. The measurements are shown in Table 1 below.

TABLE 1 Distances in the park and on the plan.

In the park	distance/ m	On Figure 6	distance/ cm
Distance between posts	2.00	Distance between posts	4.00
Width of path	1.00	Width of path	2.00
Distance of path to fence	2.00	Distance of path to fence	4.00
Distance AP			
Distance CP			

TABLE 2 Distances from fence posts to hole for flagpole/m.

AP	6.00
CP	5.00

There is a hole for a flagpole (P) in the area called 'grass'. Your task is to plot its exact position on Figure 6 by following the instructions. Your assistant (who is carrying the theodolite) hasn't arrived yet, so you use a tape to measure the distance from post A to the hole for the flagpole, P, and the distance from post C to the hole, P. You note these measurements in Table 2. Add these data to Table 1.

□ On Figure 6 what length will represent the distance AP?

■ As 2.00 cm represents 1.00 m, the distance AP is 12.0 cm.

In a similar way, the distance CP will be 10.0 cm on Figure 6. Add the values for the lengths of AP and CP on Figure 6 to Table 1.

To locate the hole for the flagpole, P, open your pair of compasses to 12.0 cm, put the point on A and draw an arc in the area called 'grass', above, about halfway between B and C. Then draw an arc of radius 10.0 cm, centred on C. These arcs should cross at the location of the hole, P. If your arcs don't cross, extend them until they do.

Join AP and CP and check with a ruler that these two sides of the triangle ACP are 12.0 cm and 10.0 cm. Triangle ACP has been uniquely defined—with sides clockwise from P, of length 10.0 cm, 8.0 cm, 12.0 cm. When all three sides are known, as in triangle ACP, a triangle can only be one size and shape.

SAQ 1 Measure the angles CAP, ACP and APC and mark the values on Figure 6.

Check your answer before proceeding. The answer to this SAQ includes a Figure (Figure 48) showing the position of the hole, P, so that you can also check the work you carried out in Guided Exercise 1.

2.2 PROPERTIES OF SIMILAR TRIANGLES

What are similar triangles? Further work on your completed Figure 6 (Figure 48 in the answer to SAQ 1), will explain. Figure 48 shows the points where the path crosses AP and CP. These have been called E, F, G and H, to give two lines, AEGP and CFHP. Mark these on your copy of Figure 6.

□ Recall the sizes of the angles in triangle ACP.

■ CAP = 56°, ACP = 82° and APC = 42°.

□ What are the sizes of the angles in triangle EFP and the lengths of its sides?

■ PEF 56°; PFE 82°; EPF 42°; EP = 7.2 cm; FP = 6.0 cm; EF = 4.8 cm.

Your measurements may differ from this, slightly—there are always uncertainties with measured data.

□ What do you notice about triangles ACP and EFP?

■ They are the same shape, that is, the sizes of the angles are the same in both triangles, but the triangles are not the same size because the sides of EFP are not the same length as the sides in ACP.

The two triangles are called **similar** triangles because the *shape* is the same, that is the corresponding angles are the same size. The side of each triangle 'facing' or opposite the corresponding angles are called equivalent or corresponding sides. So sides AC and EF are equivalent sides opposite the angle of 42°, and sides AP and EP are equivalent sides opposite the corresponding angles of 82°.

□ Can you see another triangle similar to ACP and EFP?

- Triangle GHP—check the sizes of angles of this triangle to convince yourself.

We have established that similar triangles have angles that are the same size. Now measure the lengths of the sides of triangle GHP, enter all the data in Table 3.

TABLE 3 The lengths of the sides of three similar triangles.

Triangle	line	length/ cm	line	length/ cm	line	length/ cm
ACP	AC	8.0	AP	12.0	CP	10.0
EFP	EF	4.8	EP	7.2	FP	6.0
GHP	GH		GP		HP	

Use the data in Table 3 to work out the ratios specified in Table 4.

TABLE 4 Ratios of pairs of sides in triangles.

Triangles	ratio	value	ratio	value	ratio	value
ACP/EFP	AC/EF	1.67	AP/EP	1.67	CP/FP	1.67
EFP/GHP	EF/GH		EP/GP		FP/HP	
ACP/GHP	AC/GH		AP/GP		CP/HP	

The values you should have obtained are given in Appendix 2. For a pair of similar triangles the *ratio of equivalent sides* is constant (the values across the rows of Table 4), although there maybe a small amount of uncertainty in your data. The ratio is different for different pairs, so the ratios differ down the columns as each combination of pairs is considered.

Similar triangles and their properties can be a useful device; if you can draw a small triangle, a larger similar triangle can be calculated by 'scaling up', which is what is done when building a house from a smaller-scale plan. It is the reverse of the process of map-making and plan-making, which 'scales down' from real life to a representation on a piece of paper, as our Guided Exercise did. Our plan of triangle ACP on Figure 6, is similar to the triangle ACP in the park but is 50 times smaller!

To summarize:

> The ratio of equivalent sides of similar triangles is constant.

The fact that the ratio of sides in similar triangles is constant means that if you know the lengths of *some* of the sides, the lengths of others can be calculated. Try SAQ 2 to confirm this.

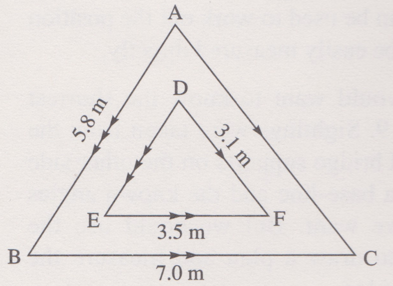

FIGURE 7 A triangular flower bed with surrounding path.

SAQ 2 A triangular flower bed has a surrounding path as shown in Figure 7. The small arrows indicate lines that are parallel. For each triangle the lengths of two sides are known. It is pouring with rain and you want to buy edging strip for the path. So that you don't get wet through, calculate the lengths of of the two unknown sides, AC and DE. Hint: what is the ratio of equivalent sides?

2.3 SURVEYING USING ANGLES

The position of the flagpole hole was plotted on Figure 6 using the *lengths* AP and CP, but surveyors also use *angles* when making plans. This Section shows you how.

Your assistant arrives and you set up the theodolite at point A and look through the telescope along the fence to C, (this is called 'taking a sighting on C'). Your assistant places a surveying post in the hole at P and the theodolite 'protractor' is

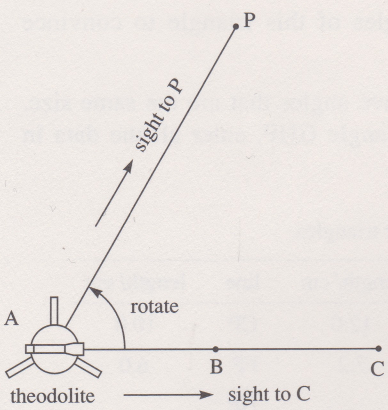

FIGURE 8 Measuring an angle with a theodolite, as seen from above.

turned horizontally to view this pole. The angle of rotation, CAP, is measured. The procedure is shown in Figure 8, as seen from above.

□ What should the theodilite measure the angle CAP to be?

■ It should be 56°, as this was the value that we measured on the plan.

The theodolite is then moved to C. By first looking back to A and then rotating to view the surveying pole at P the angle (ACP) can be measured in the park. This should be 82°, the same value as ACP on Figure 6. By drawing these angles accurately we could have located the hole for the flagpole where they cross at P. Both methods, measuring the sides or measuring the angles, have defined the position of the hole for the flagpole.

GUIDED EXERCISE 2: SURVEYING IN THE PARK, USING ANGLES

The mayor intends to plant a tree in the park and a marker stake (T) indicates where the ground staff intend to dig and prepare the hole for the tree. Your task is to put the site of the tree on Figure 6 for the borough records.

You decide to use the fenceposts at A and D as bases to take measurements. You set up the theodolite at A, sight along the fence to D and then measure the angle between the fence and the stake for the tree as 50°. This is the angle DAT. Then you move to D, repeat the procedure and measure the angle between the fence and the stake (ADT) as 80°.

On Figure 6, draw lines from A and D at the correct angles. The two lines intersect at T, the position of the stake for the tree. By this method you have located the position of the hole for the tree *without measuring the actual distance* to the tree.

To check that the site of the tree is in the correct place, do the following SAQ.

SAQ 3 Measure the distances AT and DT and write them on your copy of Figure 6. What are these distances in the park? The answer to this SAQ includes a Figure (Figure 49) showing the position of the tree, T, so that you can check the work you carried out in Guided Exercise 2.

The length that is known accurately is called a **base-line**. In this case it is the distance between the fenceposts AD. Providing one length is known, the distance to objects can be found using measured angles at each end of the baseline. So we have found one method that can be used to work out the position of an object even though the distances cannot be easily measured directly.

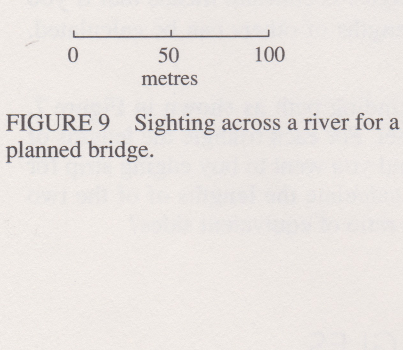

FIGURE 9 Sighting across a river for a planned bridge.

Suppose a bridge was being planned; we would want to know the shortest distance across the river, as shown in Figure 9. Sightings were taken from the ends of the base-line, A and B, to the planned bridge supports on the other side of the river. However, this method of using a base-line and the known angles would not *directly* give us the *distance* we want. But we could use the measurements of the base-line and angles to draw a plan and measure the distance on that to estimate the length of the bridge.

2.4 TRIANGLES WITH SPECIAL FEATURES

Look at the triangle ADT that you have drawn on your copy of Figure 6, (ours is shown in Figure 49). You know the sizes of two angles, as these were measured with the theodolite.

□ What is the size of the third angle, ATD?

■ 50°; because the angles of a triangle sum to 180°,
ATD = 180° − (50° + 80°).

So triangle ADT has two angles that are equal, ATD and TAD are both 50°.

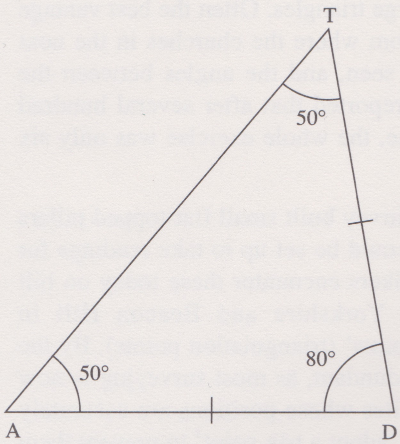

FIGURE 10 Isosceles triangle ADT, showing size of angles and tick marks.

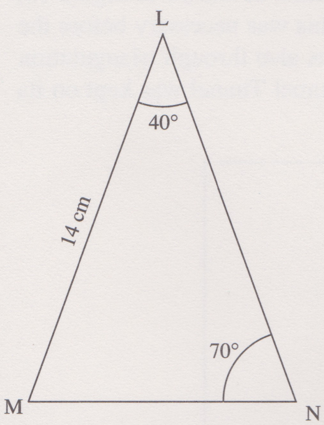

FIGURE 11 A triangle for in-text question (reduced in size).

□ What are the lengths of the sides of ADT? Write these on Figure 6.

■ AD = 12.0 cm, AT = 15.4 cm, DT = 12.0 cm.

So ADT has two equal angles and two sides equal in length; this type of triangle is called **isosceles** (meaning two equal sides). The two equal sides 'face' the two equal angles and the equal sides are usually shown by a 'tick mark', as shown in Figure 10. To check that you understand isosceles triangles, answer the following questions:

□ Look at the triangle in Figure 11. What is the angle LMN? What is the length of LN? Put these on the Figure 11. Is this triangle isosceles? If so, put 'tick marks', on the appropriate sides of the triangle.

■ LMN = 180° − (40° + 70°) = 70°. So the triangle is isosceles, with two equal angles LMN and LNM. The two equal sides are LM and LN, and these should be marked with 'ticks'. The length of LM = LN = 14 cm.

Isosceles triangles share some of the properties of equilateral triangles that were introduced in Module 9. The equilateral triangle is a special isosceles triangle, where all the sides and angles are equal, not just two sides and two angles.

The triangles you have met in *Into Science* are summarized in Figure 12.

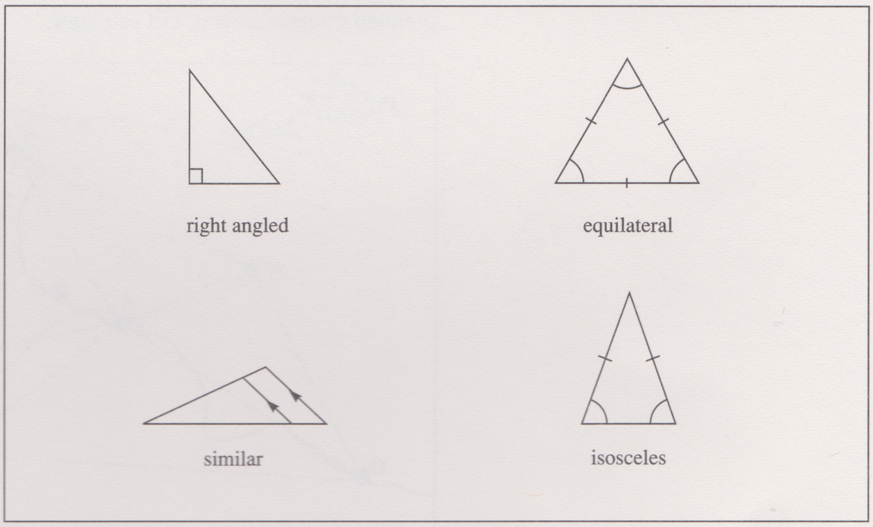

FIGURE 12 The triangles that you have met in *Into Science*.

SAQ 4 Imagine that Figure 11 has been folded in half by placing M on top of N. This creates two right-angled triangles. The fold is 13 cm long. How long is MN?

We conclude Section 2 by noting just how important this technique of 'measuring angles from each end of a base-line of known length' has been in the history of mapping and surveying, as illustrated by the Ordnance Survey.

2.5 THE ORDNANCE SURVEY

The most comprehensive survey mapping in the UK has been done by the Ordnance Survey which was originally set up in 1791 as a department of the army to provide accurate maps, mainly for military use. They used techniques of measuring distances and angles as described in Sections 2.2. and 2.3, and created maps from such a '**triangulation survey**'. The Ordnance Survey created a network of survey triangles covering the whole of the United Kingdom from an accurate baseline.

If the survey covers a large area you need large triangles. Often the best vantage point for sightings was the church tower from where the churches in the next villages or some other landmark could be seen, and the angles between the viewing lines measured accurately. It was reported that after several hundred miles of surveying and return to the baseline, the whole exercise was only six inches (15 cm) in error.

In establishing the network, the Ordnance Survey built small flat-topped pillars in prominent positions so that a theodolite could be set up to take readings for the vertices of the triangles. Walkers and hikers encounter these today on hill summits, such as Ingleborough Hill in Yorkshire and Beacon Hill in Leicestershire. The pillars are called 'trig points' (triangulation points). By the early 1990s these trig points had become redundant, as most surveying is now done from satellite photographs using satellites whose positions are accurately known. The public have been encouraged to 'adopt a trig point' to prevent them disappearing from our landscape.

It was through a triangulation survey that England and France were eventually mapped in their correct positions relative to each other, as shown in Figure 13, using the sight-lines and survey stations marked. This was necessary before the route for a Channel Tunnel could be planned. It was also through triangulation surveys *within* the tunnels that the route of the Channel Tunnel was kept on its planned course, as you will see later.

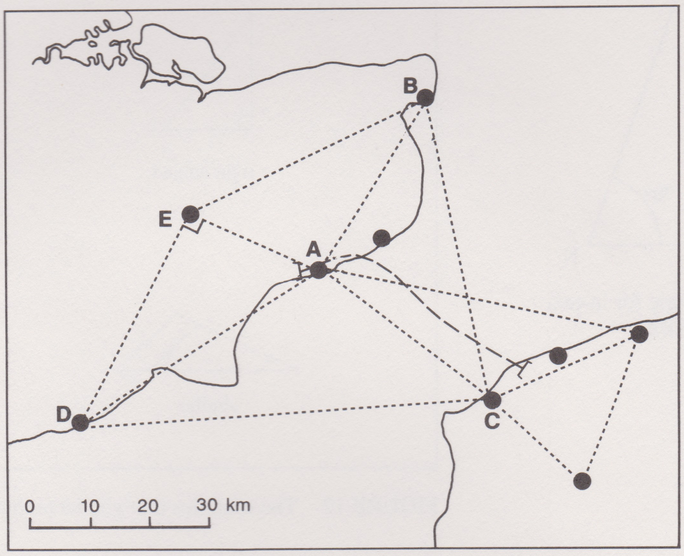

FIGURE 13 Triangulation network in SE England and across the Channel to France.

3 RIGHT-ANGLED TRIANGLES

In Section 2.2 it was established that there was a relationship between the equivalent sides of similar triangles; if some of the sides are known, others can be worked out. However there is no easy way of calculating angles from the values of the sides or vice versa. (If you have studied mathematics you will realise that there are ways to do this but these techniques are beyond the scope of *Into Science*.) This Section examines triangles that contain a right angle, in more detail. For these triangles there are simple ways of calculating lengths and angles from each other, providing us with very powerful and useful tools that are the basis of the branch of mathematics called **trigonometry**. Trigonometry, the study of the sides and angles of right-angled triangles, is used extensively in navigation and astronomy as well as surveying.

The end of this Section shows you how your knowledge of triangles can be used to complete a triangulation survey for the route of the Channel Tunnel, as shown in Figure 13.

3.1 A LITTLE REVISION AND SOME NEW TERMINOLOGY

The triangles in Figure 13 look quite complex so let's begin with a more straightforward problem. Right-angled triangles were introduced in Module 3, Section 4.5, in the context of the shape of a sloping roof. If two sides of the triangle are known the third can be calculated using Pythagoras' theorem.

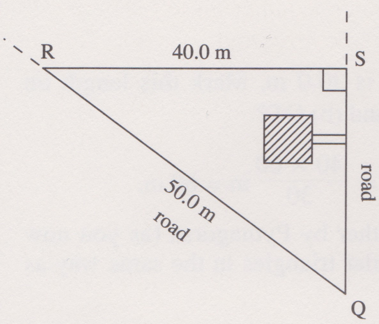

FIGURE 14 A house in a triangular plot.

☐ A house sits in a right-angled triangular plot of land QRS, with two fenced sides of length 40.0 m (RS) and 50.0 m (RQ), as shown in Figure 14. What length of fencing is needed for the third side (QS)? Write this length on Figure 14.

■ 30.0 m;
$(QS)^2 + (40.0)^2 = (50.0)^2\,m^2$

so $QS^2 = (50.0^2 - 40.0^2)\,m^2 = (2\,500.0 - 1\,600.0)\,m^2 = 900.0\,m^2$

thus $QS = \sqrt{900.0\,m^2} = 30.0\,m$.

Now look at the *angle* RQS, between the new fence and the existing one. Call this angle RQS α (alpha, pronounced as in alphabet). Mark α on Figure 14.

The Box below gives the terminology of right-angled triangles some of which was introduced in Module 3.

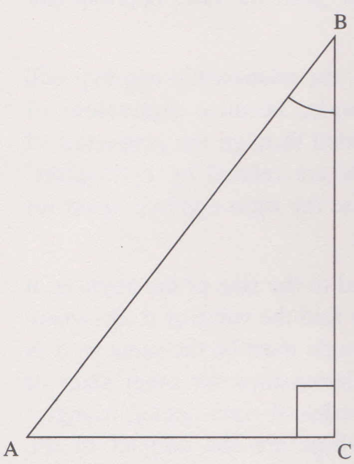

FIGURE 15 A right-angled triangle for an in-text question.

THE TERMINOLOGY OF RIGHT-ANGLED TRIANGLES

When dealing with a right-angled triangle such as the one in Figure 14, do the following:

- first locate the right angle and mark it; (this is shown in QSR)
- the hypotenuse is the side facing the right angle; hypotenuse is often shortened to hyp; write this along the line (*hyp* is QR)
- the side facing the angle marked α is called the **opposite**, shortened to opp (this is RS; write *opp* along this side)
- the side that joins α to the right angle is called the **adjacent**, shortened to adj (this is QS; write *adj* along this line)

To check that you understand the use of the new terms, try the next question.

☐ In Figure 15, which is the side *opposite* the marked angle at B? Write opposite on the correct line on Figure 15.

■ AC is opposite.

Mark *adjacent* on BC and *hypotenuse* on AB.

Now return to the triangle QRS in Figure 14. A builder buys the land, QRS and the adjoining plot, TUSR, as shown in Figure 16.

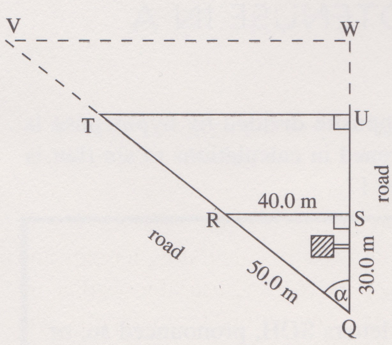

FIGURE 16 A triangular building plot, QRS, and the adjoining plot of land, TUSR.

☐ In relation to the angle α at Q in triangle QRS in Figure 16, what is the ratio:

$\frac{\text{opp}}{\text{hyp}}$ (or opp/hyp)?

■ $\frac{RS}{QR} = \frac{40}{50} = 0.8$

□ What is the relationship between triangles QRS and QTU?

■ They are similar triangles.

Recall that this means they share many properties: the angles are the same size and ratios of equivalent sides are constant.

□ The builder measures SU for fencing; it is 30.0 m. Mark this length on Figure 16. What are the lengths of (a) TU and (b) QT?

■ (a) Ratio of $\frac{QS}{QU} = \frac{RS}{TU}$ so $TU = \frac{RS \times QU}{QS} = \frac{40 \times 60}{30}\,\text{m} = 80\,\text{m}$.

(b) QT can be calculated in two ways, either by Pythagoras (as you now know the lengths of two sides), or by similar triangles in the same way as TU was calculated.

Using Pythagoras: $QT^2 = 80^2\,\text{m}^2 + 60^2\,\text{m}^2 = (6\,400 + 3\,600)\,\text{m}^2$

$= 10\,000\,\text{m}^2$ so $QT = 100\,\text{m}$.

□ What is the ratio opp/hyp for angle α in triangle QTU?

■ $\frac{TU}{QT} = \frac{80}{100} = 0.8$

So although triangle QTU is larger than triangle QRS, the ratio opp/hyp (for angle α at Q) is the same, 0.8, for both triangles.

We can go further, and say that for this size of α, the relationship opp/hyp will always be 0.8 *no matter how large the triangle*, because equivalent or corresponding sides in the two triangles are related through the properties of similar triangles, and the sides in each triangle are related by Pythagoras' theorem. You can measure the sides and calculate the ratio opp/hyp again for triangle QVW, if you are not convinced.

The converse is also true, since the ratio is related to the size of the angle α, it follows that if the ratio is known it can be used to find the value of α. So where the ratio of opp/hyp is 0.8 then the value of the angle must be the same as α in Figure 16. It is possible to find comparable relationships for other sizes of angles, because of this connection between the sides of right-angled triangles through Pythagoras' theorem. These relationships are the subject of the following subsections.

3.2 RATIO OPPOSITE/HYPOTENUSE IN A RIGHT-ANGLED TRIANGLE

The ratio examined in Section 3.1, that of opposite divided by hypotenuse is called **sine** (pronounced sign) and often shortened in calculations to sin (but is still pronounced sign). Sin α is shown in Figure 17.

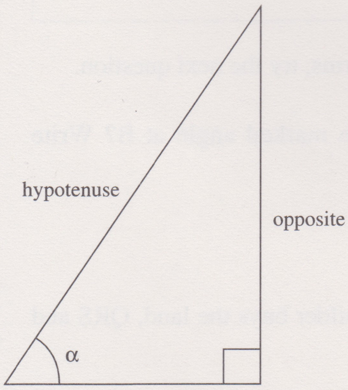

FIGURE 17 $\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$

$$\sin\alpha = \frac{\textbf{o}\text{pposite}}{\textbf{h}\text{ypotenuse}}$$

this ratio can be remembered by the initial letters SOH, pronounced so, or sew.

Sine is a ratio, a number without units, so you must always make sure that the units of any lengths involved are the same before any calculations, so that the units cancel out.

□ Look back to your Figure 6, and draw a line that goes through P and is at right angles to ABCD. Use a set square or your protractor to do this. Figure 18 shows what it should look like (Figure 18 is half the size of Figure 6 and the triangle ADT has been omitted, for clarity). For ∠RAP, measure the length of the opposite (RP) and hypotenuse (PA). What is the ratio opposite/hypotenuse?

■ 0.825. On Figure 6 the lengths are 9.9 cm/12.00 cm = 0.825. (Your result will depend upon your value for RP, if you obtained 10.0 cm, your value of sin α will be 0.833).

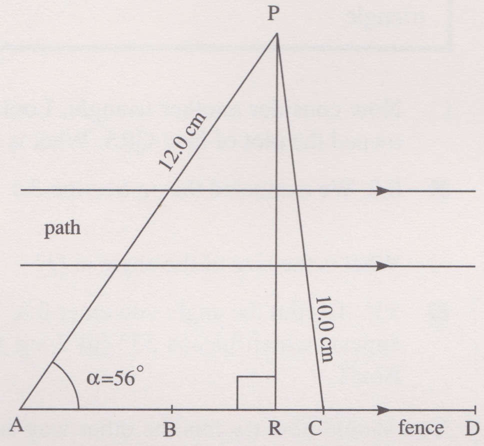

FIGURE 18 Constructing a right-angled triangle on Figure 6. Note that the size of Figure 18 is half that of Figure 6 and triangle ADT has been omitted.

We know that the angle RAP is 56°; we measured it (as angle CAP) on Figure 6 in Section 2.1 with a protractor and 'measured' it in the park with the theodolite in Section 2.3. So sin 56° = the ratio 0.825. Before calculators were available students had closely printed tables to look up the value of ratios such as the sine of an angle. Calculators make this task easy.

Evaluating sin α (to obtain the ratio $\frac{\text{opp}}{\text{hyp}}$ or opp/hyp) for the angle 56°
Press 5
press 6
press sin   0.829 037 5 will appear.

So the calculator agrees quite closely with our measured ratio.

You learned in Section 3.1 that the ratio sin is related to the size of the angle α so if the ratio is known it can be used to calculate the angle of α in degrees. Let's try doing it—finding the angle from the ratio we measured. To do this you need to use $\sin^{-1}$ (meaning 'the angle whose sin is'...) on your calculator. On some calculators the keys to press are INV sin; on other calculators they are Shift sin. If, after following the instructions given below you get an incorrect value, your calculator may not be in DEG (degrees) mode. (Consult the manufacturer's handbook.)

Evaluating α to find the angle in degrees from the known ratio

$$\frac{\text{opp}}{\text{hyp}} = 0.825$$

Enter 0.825 in your calculator.
press INV
press sin (or equivalent buttons)
55.6 will appear (to 3 sig figs). This means the angle is 55.6° or 56° to two significant figures.

To summarize:

α is an angle, measured in degrees
sin α is a ratio, of the lengths *opposite* and *hypotenuse* in a right-angled triangle

□ Now consider another triangle. Look back to Figure 16, where the builder owned the plot of land QRS. What is the sin of the angle α at Q?

■ 0.8. We evaluated this in Section 3.1

□ What is the size of the angle at Q?

■ 53°. To find the angle you enter 0.8, INV (or shift), sin and 53.130 102 will appear, which means 53° (to 2 sig figs). The keys to press are shown in Box 1.

BOX 1
Press .
press 8
press INV or shift
press sin (53.130 102 will appear)

You should now try this the other way around. Enter 53 in your calculator and press sin.

□ What is displayed?

■ 0.798 635 5 (very close to 0.8).

□ Press INV (or shift) sin (or $\sin^{-1}$). What is displayed?

■ 53, meaning 53°.

You have seen how sin α can be used to find a ratio and how this ratio can be used to find the angle in degrees. However, the formula can be used to do more than that. By rearranging the formula sin α = opp/hyp, you can find the length of the opposite when given the length of the hypotenuse and the size of the angle α, or the length of the hypotenuse when given the length of the opposite and the size of the angle α. Try this in SAQ 5.

SAQ 5 In a series of right-angled triangles, the length of the hypotenuse is 10 m. Find the length of the opposite (to 2 sig figs) when α is (a) 10° (b) 30° (c) 45° (d) 60° (e) 80°.

SAQ 6 Find (to the nearest degree) the angles whose sines are:
(a) 0.104 5 (b) 0.707 1 (c) 0.866 0 (d) 0.999 8

To summarize this Section, if you know the lengths of the opposite and hypotenuse, the angle α can be found using the ratio sine. If you know the angle and the length of one side (opp or hyp) the length of the other side can be found (hyp or opp, respectively).

Supposing you need to find the angle but know the lengths of the *adjacent* side and the hypotenuse? One approach would be to calculate the length of the opposite (using Pythagoras' theorem) and then use sine. There is an easier way using another ratio, called the cosine.

3.3 RATIO ADJACENT/HYPOTENUSE IN A RIGHT-ANGLED TRIANGLE

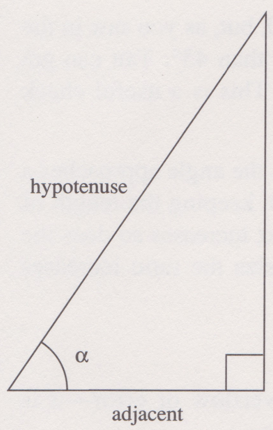

FIGURE 19 $\cos\alpha = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

The ratio of adjacent/hypotenuse (adj/hyp) is called the **cosine**, pronounced co-sign. It is abbreviated to cos, which is pronounced cos (as in cost) or coz. Cos α is shown in Figure 19. You can use this ratio in exactly the same way that you used sin α.

> $\cos\alpha = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
>
> you remember this by the initial letters CAH, pronounced ca as in cat.

□ Find the cosine of angle QRS in Figure 16.

■ From the Figure the cosine is 40/50 = 0.8

□ What is the size of the angle QRS?

■ 37°.

Angle RSQ = 90° and RQS = 53° (using sin α in Section 3.2). The third angle, QRS, must be 37° as the angles of a triangle sum to 180°. In a right-angled triangle the other two angles must add up to 90°.

You can check the 37° result using your calculator. Enter 0.8, press INV cos (or $\cos^{-1}$), 36.869 898 will appear. This means 37° (to 2 sig figs).

Angle RQS is α and we will call angle QRS, β. Mark β on Figure 16.

□ In Figure 16, what is the ratio sin α (as opp/hyp) and the ratio cos β (as adj/hyp)?

■ The ratio sin α is 40/50 and the ratio cos β is 40/50: the ratios are the same.

This is true for all right-angled triangles.

□ Use the calculator to evaluate cos 60° and sin 30°.

■ They are both 0.5

□ Use the calculator to evaluate sin 45° and cos 45°.

■ They are both 0.707 1 (to 4 sig figs).

Try it with another two angles that add to 90°. Sin α and cos β are numerically equal when α plus β equals 90°.

There are three combinations of pairs of sides in a triangle; so far we have considered two of these in right-angled triangles. We look next at the third combination—adjacent and opposite.

3.4 RATIO OPPOSITE/ADJACENT IN A RIGHT-ANGLED TRIANGLE

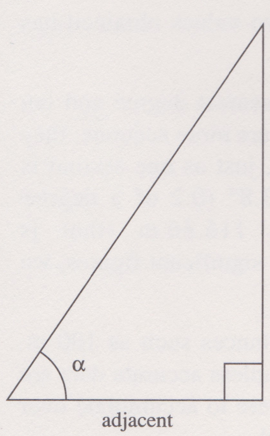

FIGURE 20 $\tan\alpha = \dfrac{\text{opposite}}{\text{adjacent}}$

There is one more ratio of sides of right-angled triangles and this is the ratio of opposite/adjacent, called **tangent** (abbreviated to tan). This is shown in Figure 20.

> $\tan\alpha = \dfrac{\text{opposite}}{\text{adjacent}}$
>
> you remember this as TOA pronounced toe–a (a as in apple)

□ Use the calculator to evaluate tan 30°, tan 45°, tan 60°, tan 89.9°

■ 0.577, 1.00, 1.73, 572.96

Sin and cos can only have values between zero and ±1.00, but, as you saw in the last question, tan is greater than 1.00 for angles larger than 45°. Tan can get extremely large as the angle approaches a right angle. This is a useful check when doing calculations.

If you want to appreciate why tan gets extremely large as the angle approaches a right angle, increase the size of the angle α in Figure 20, keeping the length of the adjacent size fixed. You should notice that as angle α increases so does the length of the opposite side. As the angle increases in size the ratio increases quite dramatically.

□ Find the tan of 90°.

■ Calculators usually display E for (exponential) overflow or error—it is trying to calculate a number so large that it can't cope.

Now that we have learnt some trigonometry we have some very powerful tools to solve problems, such as the length of the bridge shown in Figure 9. A similar situation is shown in Figure 21, where the length of sections of the base-line and the measured angles have been marked.

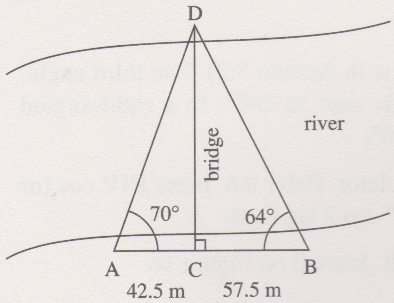

FIGURE 21 Sighting across a river for a planned bridge.

□ Look at triangle ACD in Figure 21. Which ratio will help you to find the length of the bridge between pillars on opposite sides of the river, (length CD) ?

■ Tan. $\tan 70° = \dfrac{CD}{42.5\ m}$

□ Calculate the length of the bridge.

■ 117 m (to 3 sig figs); CD = 42.5 × tan 70° (m). The keys to press are given in Box 2.

BOX 2		
Enter	42.5	
press	×	
enter	70	
press	tan	(2.747 477 4 will appear)
press	=	116.767 79 will appear

You can check the calculation using the other triangle, BCD. The ratio to use is:

$$\tan 64° = \frac{CD}{57.5\ m}$$

□ Calculate the length of the bridge using triangle BCD (that is, 57.5 × tan 64° (m)) .

■ 118 m (to 3 sig figs). The keys to press are given in Box 3.

BOX 3		
Enter	57.5	
press	×	
enter	64	
press	tan	(2.050 303 8 will appear)
press	=	117.892 47 will appear

Ooops—why do think this discrepancy between the two values obtained has arisen?

It is because the angles have been given only to the nearest degree and tan increases rapidly for large angles. Surveyors' theodolites are more accurate; they can be read to one second of arc (1/3 600 of a degree, just as one second is 1/3 600 of an hour). Recalculating using CBD = 63.8° (0.2 of a degree difference) gives the length of the bridge as 116.86 m (that is 57.5 × tan 63.8°(m)). If the original data are all to three significant figures, we can give the length of the bridge as 117 m.

This example illustrates that even over quite short distances such as 100 m, errors of one per cent (that is, 1 m in 100 m) can occur unless accurate data are measured. Imagine what would happen if such errors were to accumulate over longer distances, such as the length of the Channel Tunnel.

Some people remember the sequence of the **trigonometric ratios** (or trig ratios) of sin, cos and tan, by the initials of the sides that form the ratios:

SOH CAH TOA

Others by *pronouncing* this as 'so-ca-toa' picture it as the name of a South Sea Island, and so successfully memorize the key trigonometric ratios of sine, cosine and tangent. You must select whichever method you feel suitable—but do remember them!

3.5 SURVEYING THE POSITIONS OF ENGLAND AND FRANCE

This Section puts into practice the ratios of sides in right-angled triangles, called trig ratios. To do this we return to the Channel and the survey that established the relative positions of England and France, shown in Figure 13 and again here as Figure 22 where some of the survey stations have been labelled.

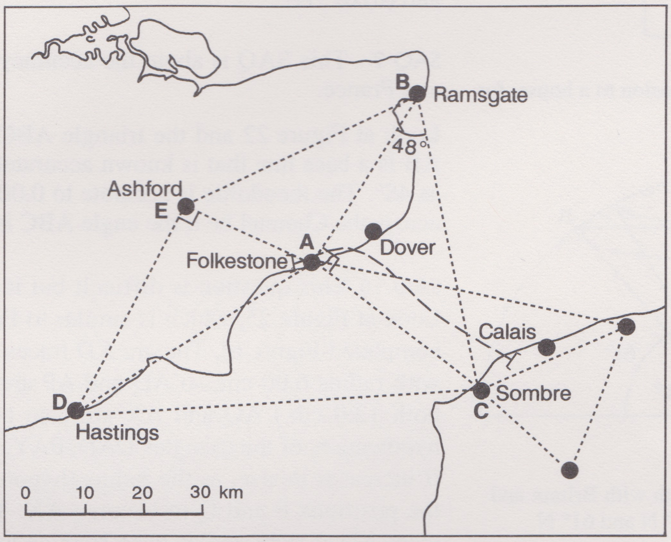

FIGURE 22 Triangulation network in SE England and across the Channel to France.

The baseline from Folkstone to Ramsgate (AB) has been established as 34 km long. A sighting from B (Ramsgate) has measured angle ABC as 48°. The angle BAC is 90°.

□ What is the distance across the Channel, that is the length AC?

■ 38 km; $\tan 48° = \dfrac{AC}{34}$ (km) so $AC = 34 \times \tan 48°$ (km)

$= 37.760\ 826$ km, or 38 km to 2 sig figs. (The keys to press are given in Box 4.)

BOX 4

Enter	34	
press	×	
enter	48	
press	tan	
press	=	(37.760 826 will appear)

Now look at triangle ADE; the distance from Folkstone to Ashford (AE) is measured as 25 km. A sighting from D (Hastings), establishes angle ADE as 34°. DEA is a right angle.

□ How far is the distance from Hastings to Folkstone (AD), that is the hypotenuse of triangle AED?

BOX 5

Enter	25	
press	÷	
enter	34	
press	sin	(0.559 192 9 will appear)
press	=	44.707 291 will appear

■ 45 km (2 sig. figs). Sin 34° = $\frac{25}{AD}$ (km) so AD = $\frac{25}{\sin 34}$ (km). (The keys to press are given in Box 5.) This is, of course, the distance 'as the crow flies'—driving or going by bus would be further.

So with a limited amount of information about the angles and distances that we started with in Figure 22, we can build up an entire network measured and calculated between the points with a high degree of accuracy.

Now try the SAQs to test your understanding of trig ratios.

SAQ 7 You are building a small extension to a house to fit beneath a first floor window, as shown in Figure 23. Building regulations insist that the roof slopes at least 35° to the horizontal. The base of the window is 3.50 m from the ground. You want the outside wall of the extension to be 2.00 m high. How far out from the house can the extension wall be (that is, the length of x in Figure 23)?

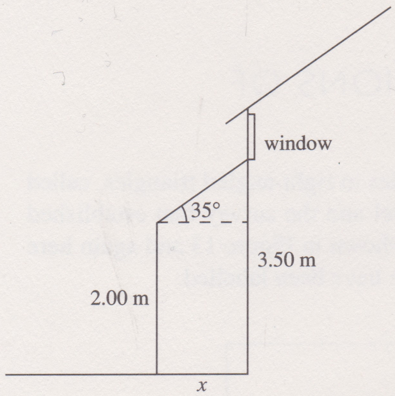

FIGURE 23 An extension to a house, for SAQ 7.

SAQ 8 The radius of the Earth is 6 370 km. Britain lies between 51° N and 61° N, as shown in Figure 24. What are the radii of the lines of latitude (a) 51° N and (b) 61° N?

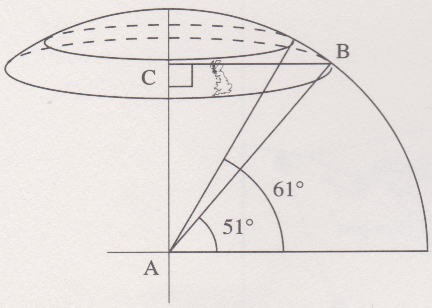

FIGURE 24 The Earth with Britain and the lines of latitude 51° N and 61° N marked, for SAQ 8.

SAQ 9 This SAQ is about the accuracy of the surface survey linking England and France.

Look at Figure 22 and the triangle ABC (Folkestone to Ramsgate to Sombre). AB is a base line that is known accurately as 34.000 0 km; ABC was measured as 48°. The theodolite is accurate to 0.000 3 of a degree. What will the distance across the Channel be if the angle ABC is 48.000 3°?

SAQ 10 This question is difficult but it shows an important property of sin α. Look at Figure 25, which is similar to Figure 18 (which is half the size of your completed Figure 6). The arc XD traces a quarter of a circle, centred on A and with radius 6.00 cm, so AD and AP are 6.00 cm. (On your Figure 6 these are both 12.0 cm.) AQ and AX are also both 6.00 cm and these radii form the hypotenuses of the triangles DAD, PAY, QAZ, XAX. You can see that the angle α increases in size, as the radius (hypotenuse) starts at AD and moves through the positions P and Q to become AX. The points at which the opposite sides meet AD at right angles have been called Y and Z. Measure the lengths of the opposite sides of the four triangles. What happens to the size of the ratio opp/hyp, in the four triangles DAD, PAY, QAZ, XAX?

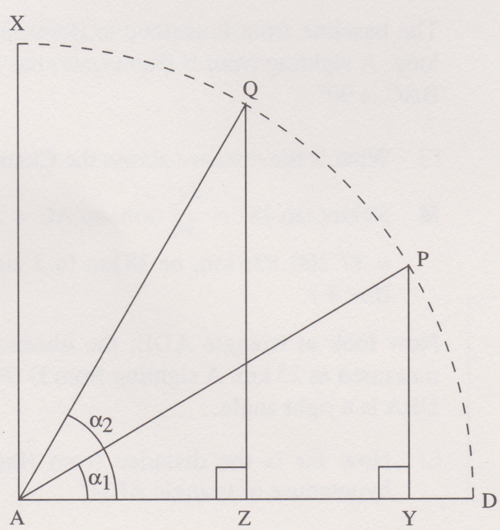

FIGURE 25 The ratio of opposite divided by hypotenuse in the four triangles DAD, PAY, QAZ and XAX.

4 SLOPES AND GRADIENTS

At this point we move away from our flat-Earth assumption and admit that the ground is rarely flat. If you walk or cycle frequently you are aware of slopes, especially steep ones that make your muscles ache. More than that, knowing how much ground slopes can be very important, in constructing roads, railways and canals, for example. To study slopes (called **gradients**), we use the trig ratio tan that you have already met.

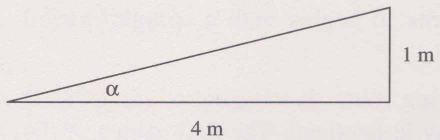

FIGURE 26 A gradient of 1 in 4.

☐ In terms of a right-angled triangle and an angle α as shown in Figure 26, what is $\tan\alpha$?

■ $\text{Tan}\ \alpha = \dfrac{\text{opposite}}{\text{adjacent}}$

If you drive or cycle you will probably have seen the term gradient of 1 in 4 or 1 in 6 as a warning sign as you approach a steep hill. These numbers give you a measure of the steepness of the hill. If you are a car driver faced with such a road sign then you would be advised to keep in low gear. An experienced cyclist could manage a 1 in 4 hill, but a less experienced one would have to get off and push. But what do these figures actually mean?

The expression 1 in 4 means that for every 4 metres measured in the horizontal direction the road rises (or falls) by 1 metre, as illustrated in Figure 26. From now on we shall use the term *run* for the horizontal distance travelled and *rise* for the corresponding vertical distance. Thus the gradient (of the hypotenuse) of the line given in Figure 26 is:

$$\text{gradient} = \frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{\text{rise}}{\text{run}} = \frac{1}{4} \text{ or } 0.25$$

$$\text{This is the same as } \tan\alpha = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{4} \text{ or } 0.25$$

So the gradient can be used to find the angle of the slope, in the same way that you can find the size of angles using $\tan\alpha$.

☐ What is the value of angle α in Figure 26.

■ 14°. $\text{Tan}\ \alpha = 0.25$, so $\alpha = 14°$ (to 2 significant figures).

Conversely, if both the size of angle α marked in Figure 26 and the distance of one side are known then the gradient can be determined.

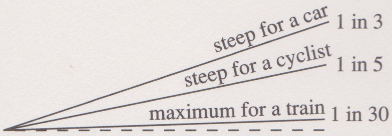

FIGURE 27 Different slopes that various forms of transport can manage.

To get a sense of the steepness of slopes or gradients, look at Figure 27 which shows the different slopes that various forms of transport can manage. Clearly a slope of 1 in 30 is quite gentle for a hiker but you may be interested to learn that the steepest slope on British Rail is 1 in 27, just outside a station in Liverpool.

☐ A car journey over the Hard Knott Pass in the Lake District can be represented by the triangle shown in Figure 28. What is the gradient of the Hard Knott Pass (to one significant figure) ?

■ $\text{Gradient} = \dfrac{\text{rise}}{\text{run}} = \dfrac{100\,\text{m}}{300\,\text{m}} = \dfrac{1}{3} = 0.3$

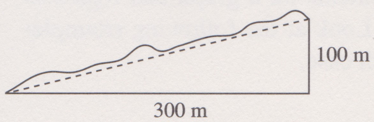

FIGURE 28 Hard Knott Pass.

☐ Notice that the gradient has no units. Why is this?

■ Metres above and below the line in the equation cancel out.

This Section has shown you how to calculate gradients. The next Section goes on to look at gradients of lines on graphs, which are calculated in the same way as we calculated the gradients in this Section.

SAQ 11 A cyclist capable of riding up a gradient of 1 in 4 plans a route through the Cotswolds which involves climbing a hill 37 metres high over a horizontal distance of 148 metres. Will she be able to ride up this slope?

4.1 INTERPRETING GRAPHS: CALCULATING SLOPES

This Section looks at plots of gradients and slopes on graphs and shows how graphs can be used to calculate gradients. It begins with a straightforward example.

Suppose you are planning an orienteering route that begins at sea level at Creetown on the south coast of Galloway in Scotland. The first stretch of the hike finishes 5 km away near the top of the hill called Cairnharrow, at a height of about 500 m. Since the fitness of the hikers is variable, you need to consider the steepness of the hill. You make a simplification that the hill has the same gradient along its length. This means that at a distance of 5 km the height is 500 m, and at 2.5 km the height is 250 m, and so on, as shown in Table 5.

TABLE 5 Vertical height above and distance from Creetown.

Distance/km	Height/m
5	500
4	400
2.5	250
1	100

□ Add Creetown to Table 5—what would be its vertical height and distance?

■ Creetown is at sea level and is thus 0 m high, and the horizontal distance from Creetown is 0 km!

We can use the data in Table 5 to plot a graph; this is shown as the solid line in Figure 29. You can see that the points lie on a straight line: this is called a **linear graph**. A linear graph shows that the relationship between the two variables (in this case height and distance) is constant, or in other words is the same all along its length. If the relationship is constant then the gradient or slope has the same steepness on every part of it. You can determine this for yourself.

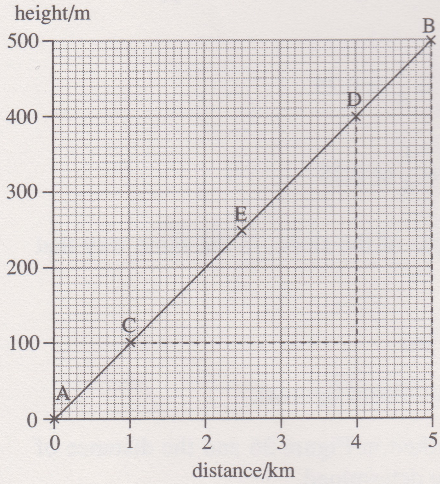

FIGURE 29 Vertical height above and distance from Creetown (using data in Table 5).

Measure the rise/run between points A and B; sketching a triangle as shown in Figure 29 might help. The vertical increase between A and B is 500 m − 0 m which equals 500 m. The horizontal increase is 5 km − 0 km which equals 5 km. Thus:

$$\frac{500\,\text{m}}{5\,\text{km}} = \frac{500\,\text{m}}{5 \times 1\,000\,\text{m}} = \frac{1}{10} \text{ or } 0.1$$

Now consider the gradient between points C and D; sketch a second triangle as shown in Figure 29. The rise is 400 m − 100 m which is 300 m. The run is 4 km − 1 km which is 3 km. Thus:

$$\frac{300\,\text{m}}{3 \times 1\,000\,\text{m}} = \frac{3\,\text{m}}{30\,\text{m}} = \frac{1}{10} \text{ or } 0.1$$

Thus the gradient of AB is the same as CD. You can see that a constant slope or gradient leads to a straight line graph. The gradient can be determined from any two points on the graph; it does not matter where you construct your triangle. Try evaluating the gradient between A and E to see if it is the same. However, it is good practice to choose points as widely separated as possible, that is to make a large triangle in order to minimize errors and to read the points on the graph easily.

Linear graphs are common in science and gradients on a graph can represent many different things, not just slopes of hills. Look at the following examples and consider what the gradient represents in each case.

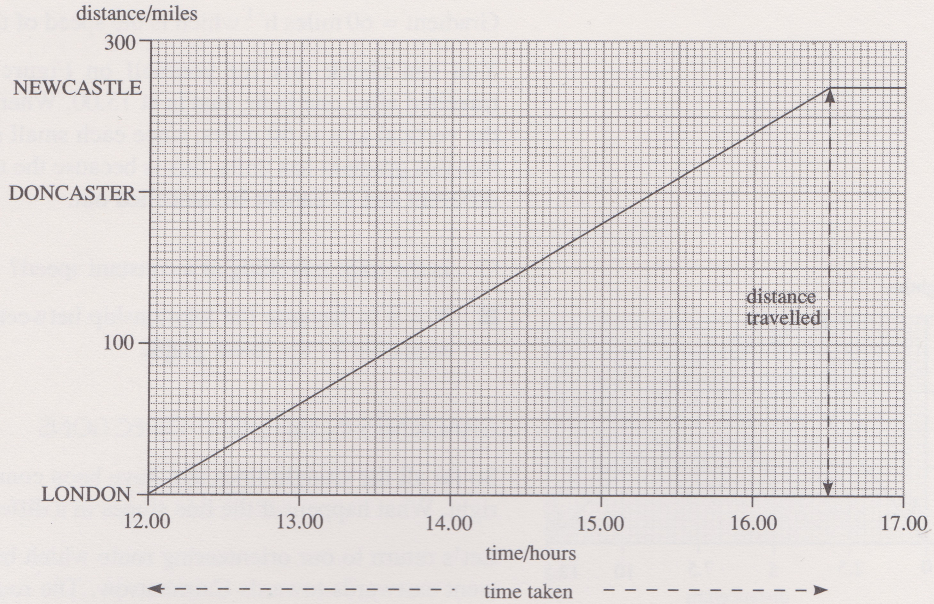

FIGURE 30 A graph representing a train journey from London to Newcastle.

The graph in Figure 30 represents a non-stop train journey from London to Newcastle. The graph shows the way in which distance varies with time. Try answering the following questions by reading from the graph.

☐ Where does the train stop?

■ The train stops at Newcastle (that is where the line of the graph becomes horizontal).

☐ At what time does the train pass through Doncaster?

■ The train reaches Doncaster at 15.21. The scale on the horizontal axis is 2 cm represents 1 hour, so one small square represents 3 minutes. To find when the train reaches Doncaster, draw a horizontal line across from Doncaster on the vertical axis until it cuts the graph; drop a vertical line to read the corresponding time on the x axis.

☐ How long does the journey from London to Newcastle take?

■ The point on the vertical axis is 270 miles, and this corresponds to 16.30 on the horizontal axis. So the journey took from 12.00 to 16.30 which is 4 hours and 30 minutes.

Now consider the gradient. What might this represent? The rise is distance and the run is time:

$$\frac{\text{rise}}{\text{run}} = \frac{\text{distance}}{\text{time}}$$

☐ Recall what is measured by distance divided by time?

■ This is a measure of speed; so the gradient in Figure 30 represents speed.

The graph can be used to give us the speed of the train. The rise (distance to Newcastle) is 270 miles − 0 miles* = 270 miles and the run (time to Newcastle) is 16.30 − 12.00 = 4 hours 30 minutes or 4.5 hours. Thus the speed of the train is:

$$\frac{270\text{ miles}}{4.5\text{ hours}} = 60\text{ miles h}^{-1}$$

* Distance travelled is still often measured in miles instead of km.

Gradient = 60 miles h^{-1} which is the speed of the train.

You can check this for yourself on Figure 30 by reading off the distance travelled after one hour, that is at 13.00. When the time is 13.00 the distance on the vertical axis is 60 miles, since each small square represents 5 minutes. Note that this gradient has units, this is because the units on the top and the bottom are different and so cannot be cancelled out.

□ Is the train travelling at a constant speed?

■ Yes it is, because the relationship between distance and time does not vary, as shown by the linear graph.

GRADIENTS IN DIFFERENT DIRECTIONS

So far all the straight lines that have been considered slope upwards from left to right. What happens if the line slopes in a different direction?

Let's return to our orienteering route which began in Creetown at sea level and went eastwards towards Cairnharrow. The route in fact continued along the top of Cairnharrow for 1 km before turning towards the coast near Cardoness Castle, 6.5 km away. The entire route is shown in Figure 31. We have already considered the line AB from Creetown to Cairnharrow above. Now look at line BC in Figure 31.

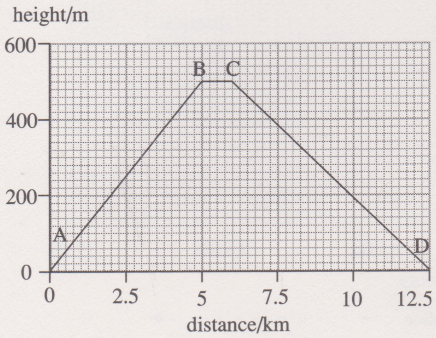

FIGURE 31 An orienteering trip from Creetown to Cardoness Castle via the Cairnharrow.

□ What is the gradient of this line?

■ $\dfrac{\text{rise}}{\text{run}} = \dfrac{0}{2} = 0$

The rise is 0 whatever run we choose; so the gradient is 0. All horizontal lines have zero gradient. Now consider the line CD in Figure 31, which represents the route from the Cairnharrow down to Cardoness Castle at sea level.

Here there is no rise but a fall of 500 m so we write: rise = –500 m. The run is 6.5 km or 6 500 m.

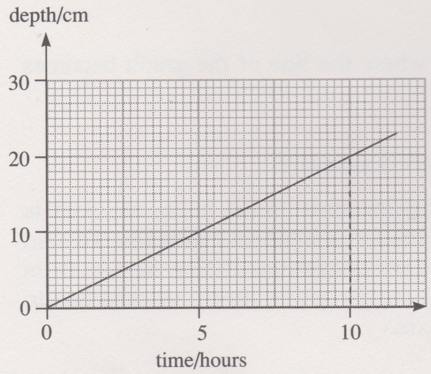

FIGURE 32 Depth of snow increasing with time.

□ What is the gradient of this slope?

■ $\dfrac{-500}{6\,500} = -0.08$ (to one significant figure)

The minus sign indicates the gradient is a fall and not a rise. So the gradient of a straight line can be either positive or negative.

The gradients you have met so far have either related to the slope of a hill or the speed of a train. However, in the following SAQ the gradient represents something different.

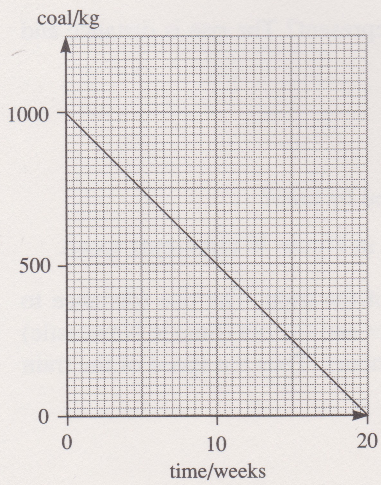

FIGURE 33 The Bright family's coal stock through winter.

SAQ 12

(a) Figure 32 shows the depth of snow with time. Measure the gradient of the graph using the triangle sketched on the Figure. What does the gradient represent?

(b) Figure 33 represents the Bright family's coal stock at the beginning of winter and the amount after 20 weeks. What is the gradient of the line in this Figure, and what does it represent?

4.2 PROPORTION

This short Section is relevant to graphs and gradients in that it is about the relationship between two variables.

A special relationship exists between distance covered and height increase as shown in Table 5 and Figure 29, and again between the length of the journey

from London to Newcastle and the time taken to make it as shown in Figure 30. Look at this relationship by examining the data, plotted in Figure 30, in more detail:

Distance of journey: 60 miles took 1 hour

(2×60 miles) = 120 miles took 2 hours

(3×60 miles) = 180 miles took 3 hours

(4×60 miles) = 240 miles took 4 hours

We describe this special relationship by saying that the time taken to make the journey is **directly proportional** to the distance travelled; as distance *increases* by a value of 60 miles then time *increases* by a value of 1 hour.

That is, distance is directly proportional to time: $d \propto t$

(where $\propto$ means 'directly proportional to')

The equation can be written $d = k \times t$ where k is a constant* called the constant of proportionality. What is the value of k in the journey from London to Newcastle?

60 miles = $k \times 1$ hour so $k = 60$ miles h^{-1}. Note that k has units.

□ If $d \propto t$ and $d = 15$ cm when $t = 5$ min, what is the constant of proportionality?

■ $k = 3$ cm min^{-1}, since 15 cm = $k \times 5$ min

Where a graph is both a straight line and passes through the origin (0, 0) as shown in Figure 30, then x and y are directly proportional, that is, as x increases by a factor (1 hour in the example above) then y increases by a constant factor (60 miles in the example above).

Another common situation is **inverse proportion**. Will the time taken to Newcastle be shorter for a fast moving train or a slow moving one? Clearly the former, a train travelling at 100 miles h^{-1} will take half the time of a train travelling at 50 miles h^{-1}; as speed *increases* the time taken *decreases*. So time taken (t) is inversely proportion to speed (s). We can write this equation:

$$t \propto \frac{1}{s} \text{ or } t = \text{constant} \times \frac{1}{s} \text{ or } t = k \times \frac{1}{s} \text{ or } t = \frac{k}{s}$$

The term *inverse* proportion is derived from the fact that 1/10 is the inverse of 10 (see Module 4).

SAQ 13 A building contractor estimates that he will complete a building in 100 days if 23 men are employed, and in 50 days if 46 men are employed.

(a) What relationship does this imply between the time taken to complete the buiding and the number of men employed?

(b) How long should the job take if 30 men were employed (to the nearest whole day).

* Constant was explained in Module 9, Section 1.3.

4.3 SYMBOLS AND EQUATIONS OF LINEAR GRAPHS

A graph clearly shows the relationship between two variables but it takes time to draw. This Section shows that the relationship between the two variables, that is the rise and run, can be described by a mathematical equation. It explains the equation but it is important that *you can use the equation to find the gradient of the graph.* The equation has the advantage that it can be written down in a fraction of the time it takes to draw a graph.

Look at the graphs in Figure 34. The axes are without units for simplicity.

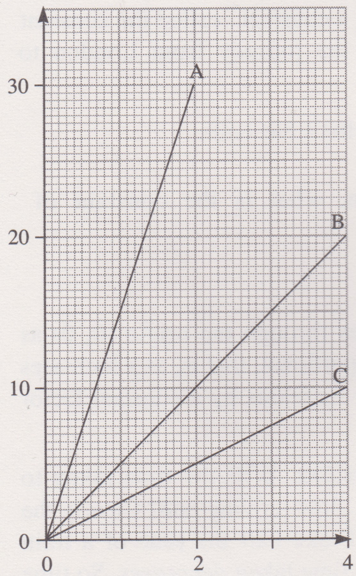

FIGURE 34 Graphs with different gradients.

□ Work out the gradients of the three graphs by drawing triangles at appropriate places.

■ For graph A: $\frac{\text{rise}}{\text{run}} = \frac{30}{2} = 15$, graph B: $\frac{20}{4} = 5$, graph C: $\frac{10}{4} = 2.5$

If you compare the gradients for the three graphs you may not be surprised at their relationship to each other. For example, the gradient of graph A is 15 which is three times steeper than the gradient of graph B which is 5. In fact, graph A looks to be three times steeper than graph B in Figure 34.

Section 4.1 showed that the relationship between rise and run of any graph is:

$$\frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \text{gradient.}$$

$$\text{For graph A: } \frac{\text{change in } y}{\text{change in } x} = \frac{y}{x} = 15$$

To make y the subject of the equation multiply both sides by x.

$$\frac{y}{x} \times x = 15 \times x, \text{ so } y = 15x$$

This is the equation for graph A. It tells us how the variables x and y for graph A are related. Most importantly, the equation for y shows the gradient as the value *before* the x. Thus: $y = \text{gradient} \times x$. This is true for all straight line graphs. Since the equation for graph A is $y = 15x$ then the value of the gradient is 15.

Now consider the equations of the graphs B and C in Figure 34.

□ The equations for graphs B and C can be derived in the same way as the equation for graph A. These are: $y = 5x$ and $y = 2.5x$ respectively, so what is the gradient for each line?

■ The gradient is 5 for graph B and 2.5 for graph C.

In science it is conventional to use the symbol m to represent gradient. This gives the general expression for a straight line graph as $y = mx$ (where the line passes through the origin that is (0, 0)).

> The general equation for a straight line passing through the origin is: $y = mx$ where m is the gradient.

Now look at another important use of the equation of a straight line graph; the equation can be used to find any point on the graph. Try one.

□ Find the value of y from the equation for graph A when $x = 1$.

■ For $y = 15x$ when $x = 1$. Then $y = 15 \times 1$ so $y = 15$.

The equation tells you that when $x = 1$ then $y = 15$. Check this on the graph by reading up from 1 on the x axis until you reach the line and then find the corresponding value on the y axis; this is 15.

□ Now use the same equation to find the value of x when $y = 20$ for graph A.

■ For $y = 15x$, when $y = 20$

$$\text{then } 20 = 15x \text{ or } \frac{20}{15} = x, \text{ so } x = 1.3$$

We can check this answer on Figure 34 by reading along from 20 on the y axis to graph A and then find the corresponding value on the x axis; this is 1.3. So the equation can also be used to find any point on the graph.

So far in Section 4.3 only straight line graphs that go through the origin have been described. In these cases x and y are directly proportional, and the equation describing the graph has the form $y = mx$. The next Section goes onto consider straight line graphs that cross the y axis at a point other than at the origin.

SAQ 14

(a) Using the equation for graph B in Figure 34, $y = 5x$. Find the value of y when x = 3. Check your answer from the graph.

(b) Using the equation for graph C in Figure 34, $y = 2.5x$. Find the value of x when $y = 7.5$. Check your answer from the graph.

(c) Give the equation for the straight line graph in Figure 30, where x is measured in hours and y in miles.

SAQ 15 What are the gradients of the following graphs?

(a) $y = 5x$ (b) $2y = 16x$ (c) $5y = x$

4.4 GRAPHS THAT DO NOT PASS THROUGH THE ORIGIN

This Section looks at gradients on graphs that do not pass through the origin (0, 0). It then goes on to describe the mathematical equation for this type of line. Again the equation has the advantage that it can be written down in a fraction of the time that it takes to draw a graph.

Consider the three graphs, A, B and C on Figure 35. The units on the axes have been omitted for simplicity.

□ What is the gradient of each of the three graphs?

■ For graph A: $m = \frac{20}{4} = 5$, graph B: $m = \frac{30 - 10}{4} = \frac{20}{4} = 5$,

graph C: $m = \frac{40 - 30}{2} = \frac{10}{2} = 5$.

If you calculated all the gradients correctly you should find that all the graphs have the same gradient of 5. In fact you can see that the graphs have the same gradient by looking at Figure 35; all the lines are parallel. However, the graphs differ from each other; how? They cross the y axis at different places—the value of y when $x = 0$ is different for each graph. This means that the equations that describe them must be different.

Begin with graph A. The equation for this line is $y = 5x$. As you saw above this is because $y = 0$ when $x = 0$.

□ In the case of graph B when $x = 0$, what is y?

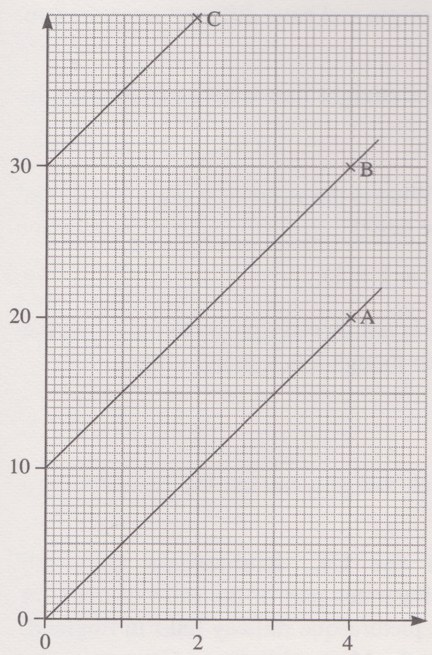

FIGURE 35 Graphs with the same gradient.

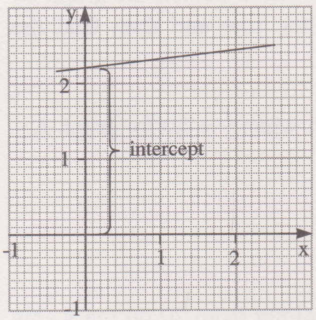

FIGURE 36 The intercept.

■ For graph B when $x = 0$ then $y = 10$.

The graph crosses the y axis at a distance called the **intercept**, shown in Figure 36. In other words the intercept is the value of y when x is zero. Thus the equation of the graph has to take account of the fact that the line passes through (0, 10). This gives the equation of graph B as: $y = 5x + 10$.

□ Check that this equation is correct by working out what y equals when x is zero.

■ $y = 5 \times 0 + 10$ which is $y = 0 + 10$ so $y = 10$.

In this equation the gradient is 5. This is independent of whether or not the line passes through the origin (0, 0). The intercept is 10.

In science it is conventional to use the symbol c to represent the intercept. This gives the general expression for a straight line graph as $y = mx + c$ where m is the gradient and c is the intercept. This equation is an important one in science.

> The general equation for a straight line is: $y = mx + c$ where m is the gradient and c is the intercept.

Check your understanding using graph C in Figure 34.

□ What is the gradient, the intercept and the equation of graph C?

■ The gradient is 5, the intercept is 30 and the equation is $y = 5x + 30$.

> The equation for a straight line graph can be used to find:
> (i) the gradient
> (ii) the intercept
> (iii) the value on one axis, given a value for a point on the other axis.

So the equations for graphs A, B and C in Figure 35 are:

Graph A: $y = 5x$, graph B: $y = 5x + 10$, graph C: $y = 5x + 30$.

These three equations show you at a glance that:

(i) the three graphs have the same gradient

(ii) graph A passes through the origin

(iii) graphs B and C do not go through the origin and have different intercepts on the y axis.

A summary of the uses of the equation of a straight line is given in the Box in the margin.

□ A graph has the equation $y = 2x + 1$. What is the value of y when $x = 0$? (In other words what is the intercept of the line?)

■ The value of $y = 1$ ($y = 2 \times 0 + 1 = 0 + 1 = 1$).

Note that the relationship between x and y is *not* directly proportional when the graph does *not* pass through the origin, even though it is a straight line.

THE LONG DRAG

The aim of this example is for you to check that you can use the equation for a straight line graph.

It is time to visit another part of the UK, Ribblesdale near the Lancashire-Yorkshire border. Figure 37 illustrates part of the railway track on the Settle to Carlisle line, one of the most picturesque railway lines in England. The route was constructed in the 1870s by the Midland Railway Company who wished to develop a fast route across the Pennine Hills to Scotland. The particular section we shall look at is the 'Long Drag', so called by the engine drivers who in the days of steam had to coax the trains up the long slope from the low-lying Ribble valley. The 'Long Drag' stretched from Settle via Horton

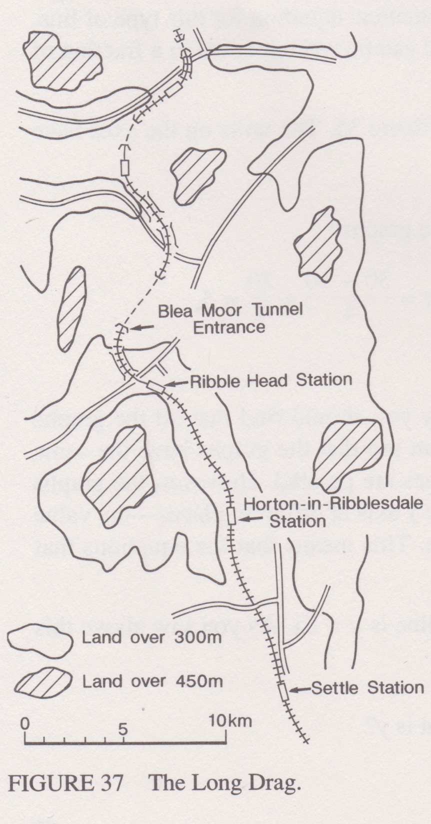

FIGURE 37 The Long Drag.

and Ribble Head station to Blea Moor Tunnel, a total horizontal distance of 21 km. Settle station is 140 m above sea level and the track rises to 350 m at the entrance to Blea Moor Tunnel. A tunnel was necessary to avoid even steeper gradients to get over into the Eden Valley and Carlisle. Although a rise of 350 m − 140 m in 21 km may sound a gentle gradient, it was a noticeable and lengthy incline for steam trains.

The height above sea level and distance from Settle to the other two stations, Horton and Ribble Head and to the Tunnel are given in Table 6.

TABLE 6 Vertical height above sea level and distance from Settle for each locations on the Long Drag.

Location	Vertical height above sea level/m	Horizontal distance from Settle/ km
Settle	140	0
Horton	240	10
Ribble Head	315	17.5
Blea Moor	350	21

The data in Table 6 are plotted on a graph in Figure 38. The x axis is distance from Settle and the y axis is vertical height above sea level.

☐ Calculate the gradient of the railway track using either the data in Table 6 or the graph in Figure 38.

■ The rise between Settle and Blea Moor is 350 m − 140 m = 210 m. The run between Settle and Blea Moor is 21 km − 0 km = 21 km.

$$\text{Thus: } \frac{\text{rise}}{\text{run}} = \frac{210\,\text{m}}{21 \times 1\,000\,\text{m}} = 0.01$$

Look at the graph in Figure 38, when $x = 0$ what is the value of y? The line crosses the y axis at 140 m.

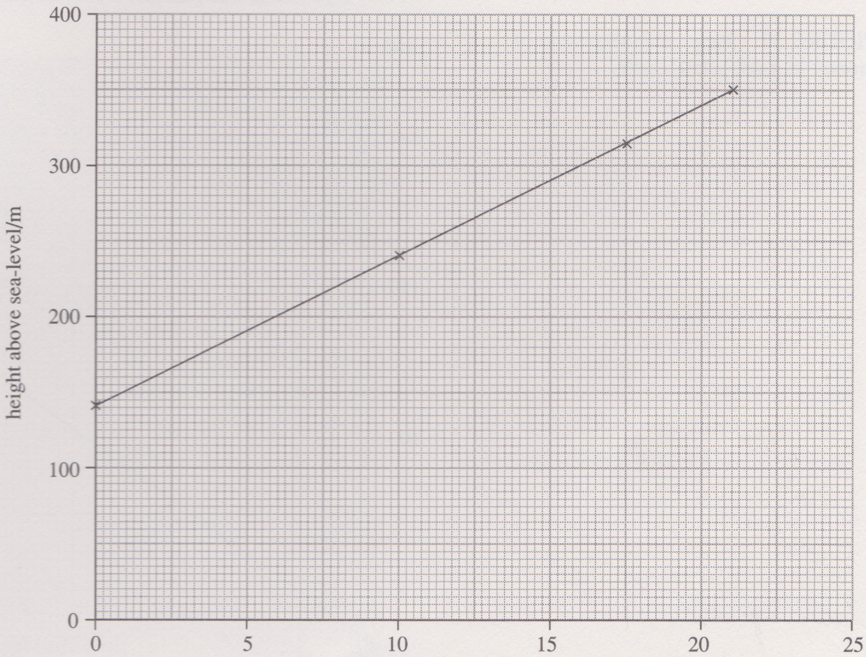

FIGURE 38 Graph of the Long Drag.

☐ What is the equation for the Long Drag?

■ $y = 0.01x + 140$.

In this equation, 0.01 is the gradient which is the way that the height varies with distance and 140 is the intercept.

☐ Check the value of y when $x = 5$ km.

■ $y = 0.01 \times 5\,\text{km} + 140\,\text{m}$ which is:
$y = 0.01 \times 5 \times 1\,000\,\text{m} + 140\,\text{m}$
So $y = 50 + 140$, or $y = 190\,\text{m}$.

Check this is correct by reading from the graph in Figure 38.

This Section has considered the equation of a straight line graph that crosses the y axis at a point other than the origin. The example of the Long Drag was concerned with heights and distances. Does the intercept always represent height? The answer to this question is a most definite "no". Try the following SAQs to convince yourself.

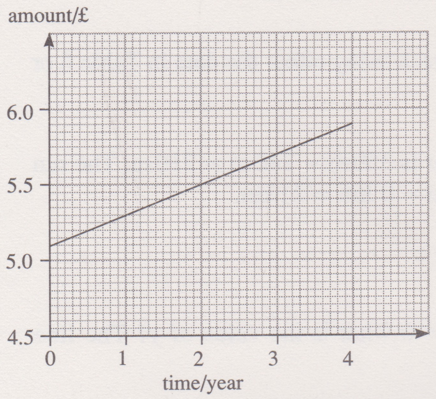

FIGURE 39 Increase in the value of money in a deposit account.

SAQ 16 The graph in Figure 39 indicates the increase in the value of a sum of money invested in a deposit account and left to accumulate interest.

(a) Find c and m in the equation for this graph: $y = mx + c$.

(b) What does the intercept represent in this graph?

(c) What does the gradient represent?

The freezing point and boiling point of water, measured in degrees Fahrenheit, are 32 °F and 212 °F respectively. On the Celsius scale these points are 0 °C and 100 °C respectively. We can plot a graph with the temperature in degrees Fahrenheit as the vertical axis, and the temperature in degrees Celsius as the horizontal axis, that is, a graph of °F and °C, by first plotting the two points (0, 32) and (100, 212) as shown in Figure 40.

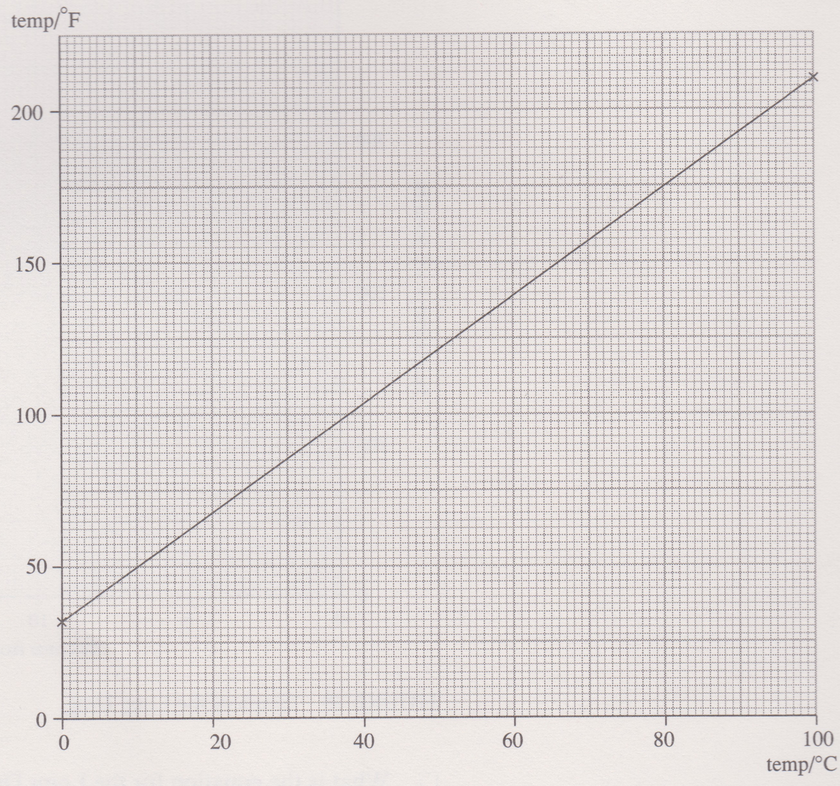

FIGURE 40 The relationship between °F and °C.

Both Celsius and Fahrenheit are linear scales. On the Celsius scale the range from °C to 100 °C is divided into equal parts, each of 1 °C, and on the Fahrenheit scale the range from 32 °F to 212 °F is divided into 180 equal parts, each of °F. So the relationship between °F and °C must be a linear one, represented by the straight line joining the two points (0, 32) and (100, 212).

SAQ 17

(a) Use the graph in Figure 40 to determine normal body temperature in degrees Celsius (assume body temperature is 98 °F).

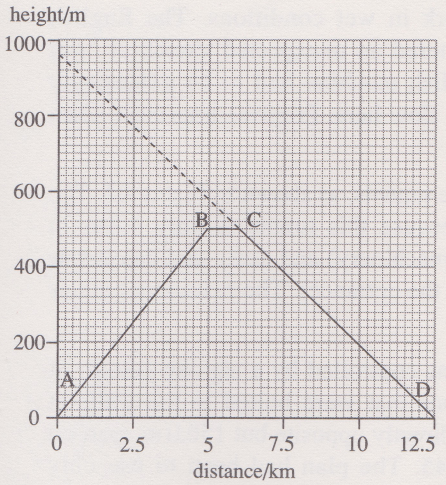

FIGURE 41 The intercept of the line CD with a negative gradient.

(b) Use the gradient and the intercept of the graph in Figure 40 to derive an equation relating degrees Celsius to degrees Fahrenheit.

SAQ 18 Look back at Figure 31. To determine the intercept of line CD, it needs to be extended backwards until it crosses the *y* axis, as shown in Figure 41.

(a) What is the intercept?

(b) What is the equation of the line CD?

SAQ 19 Figure 42 represents the walk a couple take. They leave home at 10.00 hours and return home at 19.00 hours.

(a) How fast are the couple walking in section A–B?

(b) Give an equation for the line in section B–C.

(c) Give an equation for the line in section C–D.

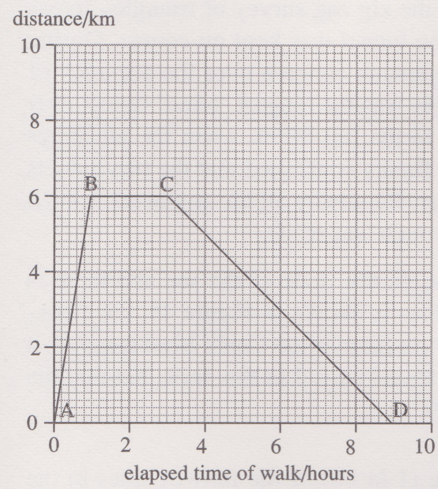

FIGURE 42 A couple's walk.

5 MEASUREMENT AND THE CHANNEL TUNNEL

Triangulation across the Channel in the early 1800s established that the length of a tunnel would have to be at least 38 km. However, it wasn't until the 1860s that the geology beneath the Channel was understood sufficiently for realistic plans to be made.

You know from the geological map of the British Isles, Figure 6 in Module 3, that Chalk is found at the surface at Dover (and also near Calais, on the other side of the Channel). Chalk is a rock that allows water to pass through, especially along natural fractures in the rock. Water is a tunneller's nightmare—it can flood workings unexpectedly, damage machines and cause death. The Chalk of the White Cliffs also contains bands of hard silica called flint that can damage tunnelling machines. Beneath the 'white' Chalk are other kinds of Chalk that have a higher content of clay (about 40%) and no flints. This 'Chalk Marl' is ideal for tunnelling because the clay content makes the rock more plastic and less likely to fracture. Also, its great advantage is that water does not pass through. (Clay was often used to line canals and ponds so that they retained water). Thus technical reasons dictated that the tunnel followed the layer of Chalk Marl, shown in Figure 43, which is about 30 m thick.

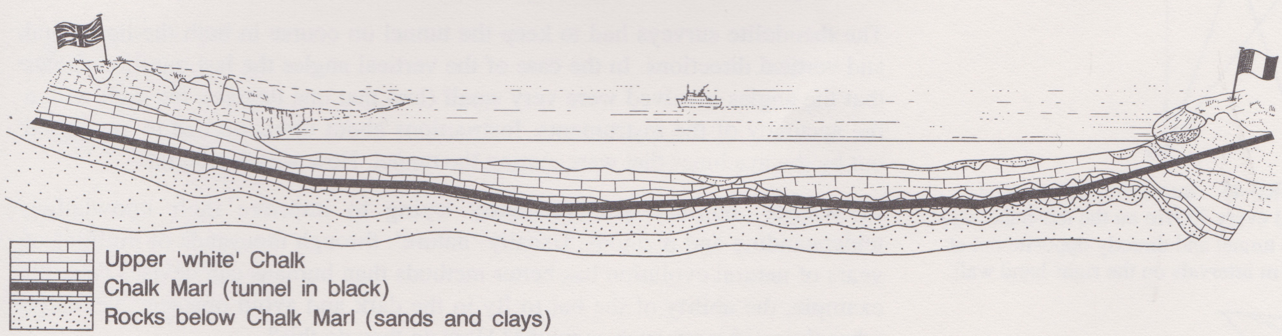

FIGURE 43 Cross-section of the English Channel, showing the geology beneath the sea-bed.

Cores were drilled from miniature drilling rigs along the line of the tunnel to confirm the geological structure beneath. Unfortunately the Chalk Marl is not flat and level; it goes up and down. So does the tunnel to stay in this geological layer. Nor is the Chalk Marl uniform in composition and nature from one side of the Channel to the other (rocks often vary considerably over quite short distances). It was known that there were major fractures within the Chalk on the French coast that would let water through, so the French cutting machines were

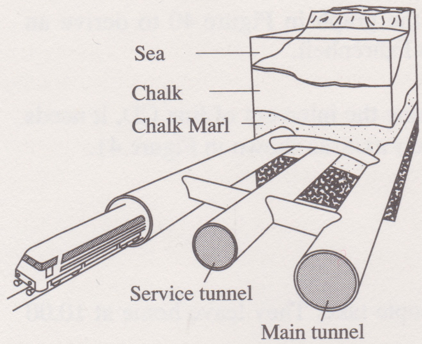

FIGURE 44 The three tunnels that make up the Channel Tunnel.

designed to be watertight, that is to work in wet conditions. The English machines were not, so when they encountered an unexpected stretch of 'wet chalk' 3 km out from the coast, the English machine had to be extensively modified. Not surprisingly, tunnelling was slow here. Progress improved when this 'wet chalk' was left behind.

Building the Channel Tunnel involved three tunnels, as shown in Figure 44. The first to be dug was a service tunnel. Because this tunnel is only 4.8 m in diameter, working in it was hot (from the machinery) and dusty and dangerous (because of frequent trains taking waste to the entrance). The two main tunnels are each 7.6 m in diameter.

In each tunnel, the direction of digging was controlled by a zig zag pattern of theodolite surveys from instrument stations that were bolted to the walls at 75.0 m intervals on the left-hand side, and directly opposite but 150.0 m apart on the right-hand side, as shown in Figure 45. The plan had been to use only stations on the left-hand wall. However, measuring through cold air close to the concrete walls compared with warm moving air in the centre of the tunnel, caused errors in the sensitive instruments, which, over the length of the tunnel and the tens of thousands of instrument readings, could have added together and caused the tunnels to miss each other. So the zig zag survey of triangles across the tunnel was necessary to ensure that the tunnel was kept on-course. During the planning and construction there was genuine concern over the accuracy of the surveys guiding the tunnels because this was the longest 'blind' tunnel ever constructed. Concern continued right up to the last 50 m, when the machines finished and manual work began. A probe finally confirmed that the two tunnels were within a few centimetres of each other. Reputedly all the engineers wanted to be in the tunnel when the final break-through was made.

On average the tunnel is about 40 m beneath the floor of the sea and there is about 60 m of water above the sea bed. This is shown in Figure 43, the cross-section of the tunnel. The variations in the position of the Chalk Marl meant that engineers had to ensure not only that the tunnel stayed in this rock, but that the gradient in any stretch did not exceed 1 : 80.

□ What is the maximum angle of slope (1 in 80) of the railway lines in the Channel Tunnels?

■ 0.72° (to 2 sig figs). Gradient is measured by the tangent, where $\tan\alpha$ = opp/adj and where opposite = 1 and adjacent = 80. Gradient = 1 ÷ 80 = 0.0125 As an angle: $\tan^{-1}$ 0.0125 = 0.72°. (The keys to press are shown in Box 6.)

BOX 6		
Press	1	
press	÷	
enter	80	
press	=	0.012 5 will appear
press	INV (or shift)	
press	tan	0.716 159 9 will appear

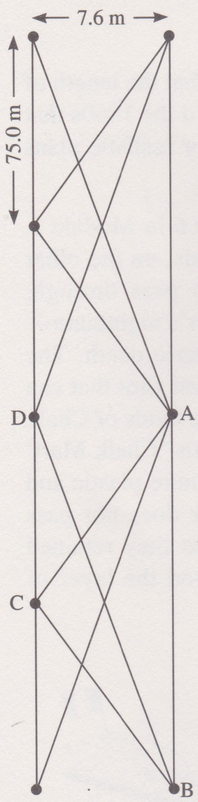

FIGURE 45 Surveying stations were placed every 75.0 m on the left-hand wall of the tunnel and directly opposite but at 150.0 m intervals on the right-hand wall.

The theodolite surveys had to keep the tunnel on course in both the horizontal and vertical directions. In the case of the vertical angles the last question shows that the angles involved were very small (less that one degree), but well within the accuracy of the instruments. Instructions about direction had to be carried out by the machines that were cutting the tunnel. How was this done?

In science, humans are constantly trying to improve their knowledge, understanding and methods. Usually 'nature', through thousands of millions of years of natural evolution has better methods than humans can devise. Take for example, the ability of the bat to fly in the dark and avoid obstacles yet catch other flying objects, such as moths? How do they do this?

Many bats use pulses of high frequency sound to navigate and detect their prey. The high frequency of the sound produces strong echoes from very small objects and by listening to the reflections the bat can differentiate between stationary objects (obstacles) to avoid them and flying objects (often prey). This is shown in Figure 46. The bat needs extremely sensitive ears to pick up the echoes, because for a distance of one metre the time between the pulse being sent out and the returning echo is just over 6 thousandths of a second (0.006 s). The sound pulses need to be very short so that outgoing pulses do not drown incoming sounds.

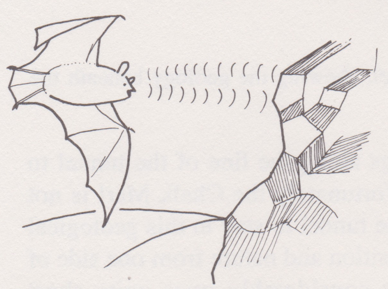

FIGURE 46 A bat sends out high frequency sound to detect objects in its path.

Humans copied this technique when developing sonar and radar systems. In the former, pulses of sound are reflected from objects to detect them (such as enemy submarines in World War 2 films). Radar uses radio waves to locate objects and is extensively used by navigational systems in boats, aircraft and in air traffic control. By making a continuous circular 'sweep', a map of the objects in an area can be seen on a display screen that is centred on the radar emitter.

To measure long distances accurately, electronic devices have been developed that use high intensity light of one wavelength. These are called lasers. When building the Channel Tunnel lasers were used in a different way, rather like a large spirit level to align the tunnelling machines. The laser light provided a thin, bright beam through the dusty conditions of the tunnel. The control zig zag survey was kept close behind the cutting machines, and the engineers on each shift used the survey to set up the laser guide for steering the machines.

Also there was a series of independent surveys along the tunnel during holiday shut-downs and maintenance shifts. These quick checks were weaker than the control survey (the zig zag), but had the advantage of being completely independent. Such was the concern over the accuracy of boring blindly beneath the Channel, with the ever-present worry that the tunnels might not meet on line. In the event, they achieved their aimed-for accuracy of being within 500 mm horizontally and 200 mm vertically. This was quite an achievement considering that the points from which measurement began were 38 km apart.

SAQ 20 Look back to Figure 45 which shows the zig zag survey in one of the main tunnels, and consider the three stations forming the triangle ADC. The two survey stations A and D are directly opposite each other, so making a right angle with the sides of the tunnel, but there are half as many stations on the right-hand wall. Using the distances given in the Figure, what is the sighting angle of A from C (that is, the angle ACD) ?

SAQ 21 An aircraft begins its descent to an airport from a height of 5 000 m, flying at a descent angle of 15° as shown in Figure 47. At 2 000 m the aircraft continues level flight. How far did the aircraft travel during the descent?

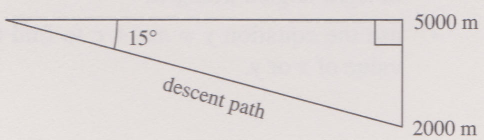

FIGURE 47 Descent path of aircraft, for use with SAQ 21.

SAQ 22 This is to brush up your algebra, as well as trig.

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}},\ \cos\alpha = \frac{\text{adj}}{\text{hyp}} \text{ and } \tan\alpha = \frac{\text{opp}}{\text{adj}}$$

Express $\tan\alpha$ only in terms of $\sin\alpha$ and $\cos\alpha$.

6 OVERVIEW

There are no new important scientific concepts that you need to remember, though the mathematical skills have been introduced within a scientific context. The main purpose of this Module is to develop mathematical skills associated with trigonometry, geometry and algebra (the equation of a straight line).

SUMMARY

These are the concepts you have learned about in this Module:

- Similar triangles are the same shape, but different sizes.
- Isosceles triangles have two equal sides and two equal angles.
- The sides of a right-angled triangle are called hypotenuse, opposite and adjacent.
- The relationships between the sides of right-angled triangles are called sine (sin), cosine (cos) and tangent (tan) and these can be used to find unknown sides or angles.
- The ratios, sin α and cos β are numerically equal if $\alpha + \beta = 90°$.
- The slope of a line is the tangent of the angle of slope.
- The sequence of trigonometric ratios can be remembered by the mnemonic SOH CAH TOA.

SKILLS

At the end of this Module you should be able to:

- use the ratios of sides to find the lengths of sides in similar triangles
- use the relationships sin, cos and tan to obtain angles or the lengths of sides of right-angled triangles
- use the equation $y = mx + c$ to find the slope of a line, the y intercept, or a value of x or y.

APPENDIX 1: EXPLANATION OF TERMS USED

ADJACENT (side of a triangle) This is the side joining the right angle and the angle in question.

BASE-LINE This is an accurately known length between two points.

COSINE (cos) The ratio of adjacent to hypotenuse:

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$

DIRECT PROPORTION Two quantities are directly proportional to each other if an increase by a factor in one of them is accompanied by an increase by a constant factor in the other. If the quantity y is directly proportional to x, their relationship is written as $y \propto x$. If a graph is both a straight line and passes through the origin (0, 0) then x and y are directly proportional.

GRADIENT A measure of the steepness of a slope:

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$$

The vertical rise divided by the horizontal distance travelled.

INTERCEPT The value on the y axis of a graph where the line crosses it.

INVERSE PROPORTION Two quantities are inversely proportional to each other if an increase by a factor in one of them is accompanied by a decrease in the other by a constant factor. If the quantity y is inversely proportional to x, their relationship is written $y \propto 1/x$.

ISOSCELES A triangle which has two equal sides facing two equal angles.

LINEAR GRAPH A straight line graph with the same gradient along its entire length; it shows that the relationship between the two variables is constant or proportional.

OPPOSITE (side of a triangle) The side facing an angle in a right-angled triangle.

PLAN An accurate representation of an area. It is a map of a small area. Plans can be viewed in the Planning Department of your local council.

SIMILAR (triangles) These have corresponding angles that are the same size: they are triangles with identical shape.

SINE (sin) The ratio of opposite divided by hypotenuse:

$$\sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

SURVEY This is the process of measuring distances and angles in order to locate features accurately and put these on a map or plan.

TANGENT This is the ratio of opposite to adjacent:

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}}$$

TRIANGULATION SURVEY A method of surveying using triangles (measured angles and sides).

TRIGONOMETRY The study of angles and lengths in right-angled triangles.

TRIGONOMETRIC RATIOS These are the ratios sine, cosine and tangent (see separate definitions).

APPENDIX 2

Completed Table 3 is given in Table 7. Note, your measurements may be sightly different from ours, so your ratios may differ slightly.

TABLE 7 Completed Table 3 (Distance in the park and on the plan).

Triangle	Line	Length/ cm	Line	Length/ cm	Line	Length/ cm
ACP	AC	8.0	AP	12.0	CP	10.0
PEF	EF	4.8	EP	7.2	FP	6.0
PGH	GH	3.2	GP	4.8	HP	4.0

TABLE 8 Completed Table 4 (Ratios in triangles).

Triangles	Ratio	Value	Ratio	Value	Ratio	Value
ACP/EFP	AC/EF	1.67	AP/EP	1.67	CP/HP	1.67
EFP/GHP	EF/GH	1.5	EP/GP	1.5	FP/HP	1.5
ACP/GHP	AC/GH	2.5	AP/GP	2.5	CP/HP	2.5

SAQ ANSWERS AND COMMENTS

SAQ 1 CAP = 56°, ACP = 82° and APC = 42°. The answers you get will depend on the precision of your drawing. You may be within half a degree of our answer; you answers may differ by up to a degree. The angles are shown in Figure 48.

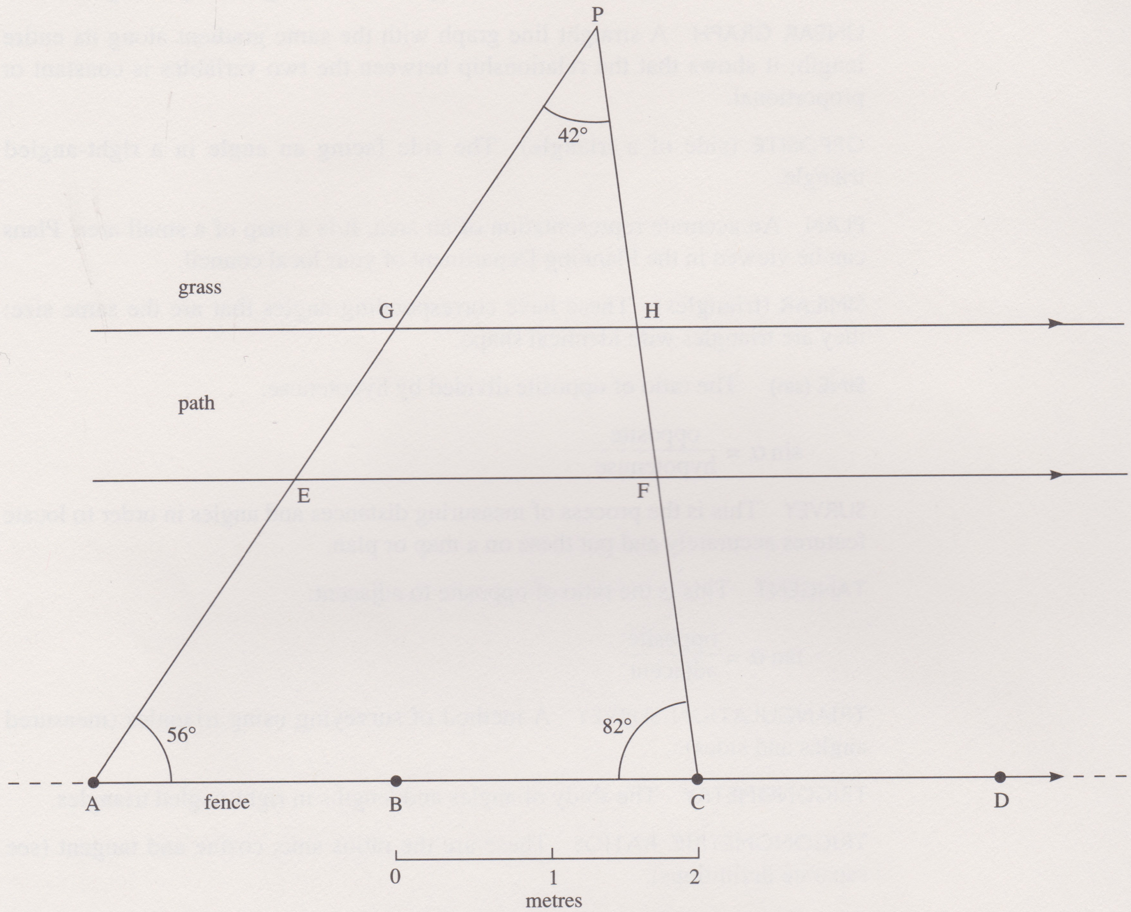

FIGURE 48 Completed Figure 6 with P (for flagpole) E, F, G, and H marked.

SAQ 2 The ratio of equivalent sides can either be expressed as

$$\frac{BC}{EF} = \frac{7.0}{3.5} = 2.0 \text{ or } \frac{EF}{BC} = \frac{3.5}{7.0} = 0.5$$

Using the factor 2.0 is simpler—the lengths of the larger triangle are twice those of the inner triangle.

$$\text{So } \frac{AC}{DF} = 2 \text{ or } \frac{AC}{3.1\,\text{m}} = 2 \text{ so AC} = 2 \times 3.1\,\text{m} = 6.2\,\text{m}.$$

$$\text{Similarly } \frac{AB}{DE} = 2 \text{ or } \frac{5.8\,\text{m}}{DE} = 2 \text{ so DE} = \frac{5.8\,\text{m}}{2} = 2.9\,\text{m}.$$

You should get the same answers using the ratio that the sides of the inner triangle are half the outer triangle:

$$\frac{DF}{AC} = \frac{1}{2} \text{ so } \frac{3.1\,\text{m}}{AC} = \frac{1}{2} \text{ or AC} = 2 \times 3.1\,\text{m} = 6.2\,\text{m}.$$

SAQ 3 Figure 49 shows the position of the tree. Note that this Figure is half the size of Figure 6. The distances on your Figure 6 should be: AT: 15.4 cm, DT: 12.0 cm. In the park the distance AT will be 7.7 m. In the park DT will be 6.0 m.

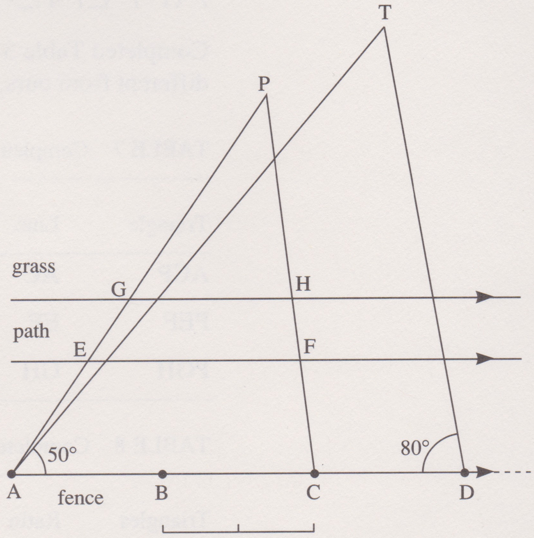

FIGURE 49 Completed Figure 6 (but reduced to half size) showing the position of the tree.

SAQ 4 This SAQ illustrates the principle that an isosceles triangle can be divided into two equal right-angled triangles. In other words, a line that cuts the non-equal angle in two meets the non-equal line at a right angle and it cuts this line in two. This is shown in Figure 50, which is similar to Figure 36 in Module 9.

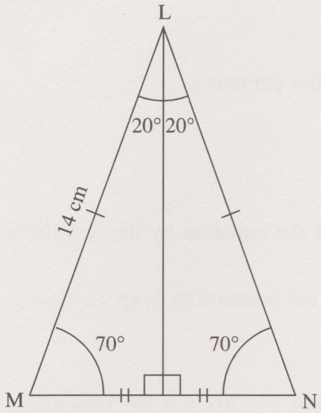

FIGURE 50 The triangle in Figure 11 cut in two.

From Pythagorus, $LN^2 = (\text{fold})^2 + \left(\frac{1}{2}MN\right)^2$

ie $14^2\,cm^2 = 13^2\,cm^2 + \left(\frac{1}{2}MN\right)^2 cm^2$

so $\left(\frac{1}{2}MN\right)^2 cm^2 = 14^2\,cm^2 - 13^2\,cm^2$

$\frac{1}{2}MN = 5.20$ cm (to 3 sig figs), so MN = 10.4 cm (to 3 sig figs).

The keys to press are given in Box 7.

BOX 7		
Enter	14	
press	x^Y	
press	2	
press	–	(196 will appear)
enter	13	
press	x^Y	
press	2	
press	=	(27 will appear)
press	√	(5.196 152 4 will appear)

SAQ 5

(a) 1.7 m (b) 5.0 m (c) 7.1 m (d) 8.7 m (e) 9.8 m

(a) The relationship is:

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}} \text{ so opposite} = \sin\alpha \times \text{hyp(m)}.$$

For $\alpha = 10°$ opposite = 0.173 648 1 × 10 m = 1.7 m (to 2 sig figs). The keys to press for (a) are given in Box 8.

BOX 8		
Enter	10	
press	sin	0.173 648 1 will appear

(b) Similarly for $\alpha = 30°$ opposite = 0.5 × 10 m = 5.0 m (sin 30° = 0.5).

(c) For $\alpha = 45°$ opposite = 0.707 106 7 × 10 m = 7.1 m (to 2 sig figs).

(d) For $\alpha = 60°$ opposite = 0.866 025 4 × 10 m = 8.7 m (to 2 sig figs).

(e) For $\alpha = 80°$ opposite = 0.984 807 7 × 10 m = 9.8 m (to 2 sig figs).

SAQ 6

(a) 6° (b) 45° (c) 60°(d) 89° The keys to press for (a) are given in Box 9.

BOX 9		
Enter	0.104 5	
press	INV (or shift)	
press	sin	5.998 360 2 will appear

SAQ 7 2.14 m

Opposite = 3.50 m − 2.00 m = 1.5 m so

$$\tan 35° = \frac{1.5}{x} \text{ so } x = \frac{1.5}{\tan 35°}\text{ (m)} = 2.14\text{ m}.$$

SAQ 8

(a) 4 010 km (to 3 sig figs).

(b) 3 090 km (to 3 sig figs), so there is a 20 km difference between the radii of the two lines of latitude.

(c) The solution is completed below for 51° N, repeat the procedure for 61° N. BC is the radius at 51°N and ABC is also 51° in triangle ABC,

$$\cos 51° = \frac{BC}{6370\text{ km}}$$

so BC = 6 370 × cos 51° km = 4 008.77 km

= 4 010 km to 3 sig figs. The keys to press are given in Box 10.

BOX 10		
Enter	6 370	
press	×	
enter	51	
press	cos	(0.629 320 3 will appear)
press	=	4 008.770 9 will appear

Alternatively, in triangle ABC, the angle BAC = (90° − 51°) = 39°;

$$\sin 39° = \frac{BC}{6\,370}\text{km}$$

so BC = 6 370 × sin 39° = 4 008.77 km

$$\text{or at } \sin(90° - 51°) = \frac{BC}{6\,370}\text{km},$$

knowing that sin (90° − 51°) = cos 51° takes us back to the first solution.

This question and answer illustrates that with trig questions there are often several ways of obtaining a solution.

SAQ 9 37.761 km

$$\tan ABC = \frac{AC}{AB}$$

so AC = AB tan 48 000 3°

= 34 × tan 48.000 3 km = 37.761 223 km.

The keys to press are given in Box 11.

BOX 11		
Enter	34	
press	×	
enter	48.000 3	
press	tan	(1.110 624 2 will appear)
press	=	37.761 223 will appear

The distance AC using ABC = 48° was 37.760 826 km. The difference between the two distances (0.000 397 km, or about 40 cm) is an estimate of the horizontal uncertainty, based on one theodolite measurement.

SAQ 10 In a triangle DAD (where $\alpha = 0°$), as the size of the opposite is zero) the ratio opp/hyp is zero (0/6). In triangle PAY the ratio is about 0.5 (3/6); in triangle QAZ the ratio is about 0.8 or 0.9 (5.2/6) and in triangle XAX the ratio is 1.0 (6/6), where the opposite overlaps and is the same length as the hypotenuse ($\alpha = 90°$). So the ratio increases from 0 to 1 as α increases from 0° to 90°, which illustrates an important property of sin α.

SAQ 11 Yes, because the gradient is $\frac{37}{148} = \frac{1}{4}$

SAQ 12

(a) Gradient is $\frac{\text{rise}}{\text{run}} = \frac{20\,\text{cm}}{10\,\text{h}} = 2\,\text{cm}\,\text{h}^{-1}$.

(Remember the units!). The gradient represents the *rate* at which the depth of snow is increasing.

(b) Gradient is:

$$\frac{-1\,000\,\text{kg}}{20\text{ weeks}} = -50\text{ kg per week.}$$

(Did you remember the units?)

This represents the rate that the coal is being used.

SAQ 13

(a) The fact that building time is halved if the number of men is doubled suggests time taken is inversely proportional to the number of men:

$t \propto \frac{1}{n}$ (where t stands for time and n for number of men) or $t = \frac{k}{n}$

(b) Substituting the figures given,

$$50 = \frac{k}{46}$$

so k = 2 300 man days.

We can use this to find the time for 30 men:

$t = \frac{2\,300}{30}$ or 76.66 days which is 77 days

(to the nearest whole day).

SAQ 14

(a) $y = 5x$ which is $y = 5 \times 3$ so $y = 15$ when $x = 3$

(b) 7.5 = 2.5x which is:

$$\frac{7.5}{2.5} = x$$

so $x = 3$ when $y = 7.5$

(c) y = 60x (rise/run = 60 miles per hour).

SAQ 15

(a) 5.

(b) 8 (Make y the subject of the equation by dividing both sides by 2).

(c) 0.2 (Divide both sides of the equation by 5, so $y = 0.2x$.)

SAQ 16

(a) $y = 0.2x + 5.10$ (If you got y = 20x + 5.10, remember when constructing an equation to make sure that the units are the same.)

The rise is £5.90 − £5.10 = £ 0.80 and the run is 4 − 0 = 4 yr

$$\frac{\text{rise}}{\text{run}} = \frac{£0.80}{4\text{yr}} = 20 \text{ pence per yr or 0.2 pounds per year.}$$

y = 0.20x + 5.10 looks clumsy. To tidy up multiply by 10:

$$10y = 2x + 51$$

(b) The intercept represents the original sum deposited.

(c) The gradient is the interest accumulating with time that is the rate of increase of the interest.

SAQ 17

(a) From the graph 98 °F is about 37 °C.

(b) From the graph, the gradient is:

$\frac{212 - 32}{100} = \frac{9}{5} = 1.8$ the y intercept is 32 °F, so

F = (1.8x + 32) C

(where x is the value of the temperature in °C that you want to change).

SAQ 18

(a) You can see that the intercept is about 960 m.

(b) $y = -0.08x + 960$

(Gradient is $\frac{500\,\text{m}}{6\,500\,\text{m}} = -0.08$)

SAQ 19

(a) 6 km h^{-1}.

(b) The couple have stopped for 2 hours; the gradient is 0, the intercept is 6 km therefore the equation is: $y = 0 \times 6 + 6$ so y = 6 km.

(c) The gradient is:

$\frac{-6\,\text{km}}{6\,\text{h}} = -1\,\text{km}\,\text{h}^{-1}$ (Don't forget the units!)

The intercept is 9 km (when $x = 0$). To find this you need to extend the line in this section backwards to the y axis. The equation for the line in this section is: $y = -1x + 9$ or $y = -x + 9$ which can be written $y = 9 - x$.

SAQ 20 5.79°

The survey stations are opposite each other, so angle CDA is a right angle, and the following ratio:

$$\tan ACD = \frac{AD}{CD} \text{ will solve the problem:}$$

AD = 7.60 m, the diameter of the tunnel.

CD is = 75.0 m, that is the distance between survey stations.

$$\tan ACD = \frac{AD}{CD} = \frac{7.60\,\text{m}}{75.0\,\text{m}} = 0.101\,333\,3$$

So, ACD = $\tan^{-1}$ 0.101 333 3 (the angle whose tangent is 0.101 333 3) = 5.786 221 1° = 5.79° The keys to press are given in Box 12.

You will see that the sighting angle is actually very small—the Figure greatly exaggerates the angle by fore-shortening the distance between the survey stations considerably.

BOX 12

Enter	7.6	
press	÷	
enter	75	
press	=	0.101 333 3 will appear
press	INV	
press	tan	5.786 221 1 will appear

SAQ 21 11.59 km, or 11 590 m

$$\sin 15° = \frac{(5\,000 - 2\,000)\text{m}}{\text{descent path}}$$

$$\text{so descent path} = \frac{3\,000\,\text{m}}{\sin 15°} = 11\,591.11\text{ m or } 11.59\text{ km.}$$

The keys to press are given in Box 13.

BOX 13

Enter	3 000	
press	÷	
enter	15	
press	sin	(0.258 819 will appear)
press	=	(11 591.11 will appear)

SAQ 22 If you had to look at the answer to get a hint, don't worry it is quite a difficult question.

$$\sin\alpha = \frac{\text{opp}}{\text{hyp}};\ \cos\alpha = \frac{\text{adj}}{\text{hyp}}$$

so opp = hyp × sin α, adj = hyp × cos α

$$\tan\alpha = \frac{\text{opp}}{\text{adj}} \text{ so substituting the values above:}$$

$$\tan\alpha = \frac{\text{hyp} \times \sin\alpha}{\text{hyp} \times \cos\alpha} = \frac{\sin\alpha}{\cos\alpha}$$